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Positive entire stable solutions of inhomogeneous semilinear elliptic equations

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ABSTRACT

For $n \geq 3$ and $p > 1$, the elliptic equation $\Delta u + K(x)u^p + \mu f(x) = 0$ in $\mathbf{R}^n$ possesses a continuum of positive entire solutions, provided that (i) locally Hölder continuous functions K and f vanish rapidly, for instance, $K(x), f(x) = O(|x|^l)$ near ∞ for some $l < -2$ and (ii) $\mu \geq 0$ is sufficiently small. Especially, in the radial case with $K(x) = k(|x|)$ and $f(x) = g(|x|)$ for some appropriate functions k, g on $[0, \infty)$, there exist two intervals $I_{\mu,1}$, $I_{\mu,2}$ such that for each $\alpha \in I_{\mu,1}$ the equation has a positive entire solution u_α with $u_\alpha(0) = \alpha$ which converges to $l \in I_{\mu,2}$ at ∞, and $u_{\alpha_1} < u_{\alpha_2}$ for any $\alpha_1 < \alpha_2$ in $I_{\mu,1}$. Moreover, the map α to l is one-to-one and onto from $I_{\mu,1}$ to $I_{\mu,2}$. If $K \geq 0$, each solution regarded as a steady state for the corresponding parabolic equation is stable in the uniform norm; moreover, in the radial case the solutions are also weakly asymptotically stable in the weighted uniform norm with weight function $|x|^{n-2}$.

1. Introduction

We consider the elliptic equation

$$\Delta u + K(x)u^p + \mu f(x) = 0 \tag{1.1}$$

and its entire solutions, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator. By an entire solution of (1.1) we mean a positive weak solution of (1.1) in $\mathbf{R}^n$ which satisfies (1.1) pointwise in $\mathbf{R}^n \setminus \{0\}$. Throughout this paper we assume that

Assumption 1.1. $n \geq 3$, $p > 1$, $\mu \geq 0$ is a constant, and K, f are locally Hölder continuous functions in $\mathbf{R}^n \setminus \{0\}$.

Liouville's theorem says that a bounded entire solution to the harmonic equation $\Delta u = 0$ in $\mathbf{R}^n$ must be constant. As we consider (1.1) as a nonlinear and inhomogeneous perturbation of harmonic equation, we expect similar behavior. The simplest case is when K and f have compact support or vanish rapidly near ∞, which for the homogeneous equation with $f = 0$, usually means that

$$K(x) = O(|x|^l), \tag{1.2}$$

at ∞ for some $l < -2$. Under this assumption, Ni in [1] studied

$$\Delta u + K(x)u^p = 0, \tag{1.3}$$

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and found infinitely many positive entire solutions which converge to positive constants at ∞. More precisely, these are countably many separated solutions by his construction in the proof. Furthermore, combining this fact and Lemma 2.5 in [2], one may conclude that if K is a radially symmetric nonnegative function satisfying (1.2), then there exists $\alpha^* \in (0, \infty]$ such that for each $0 < \alpha < \alpha^*$, (1.3) has a positive radial entire solution u_α with $u_\alpha(0) = \alpha$ satisfying

$$u_{rr} + \frac{n-1}{r} u_r + K(r) u^p = 0, \tag{1.4}$$

where $u(x) = u(|x|)$ and $r = |x|$; moreover, $0 < u_\alpha < u_\beta$ for any $0 < \alpha < \beta < \alpha^*$.

However, the existence of a continuum of positive entire solutions for (1.1) under the presence of inhomogeneous term f is still an open question. Meanwhile Naito [3] considered the non-radial case under certain assumptions on K and established that there exists a positive constant δ such that for every $l \in (0, \delta)$, (1.3) has a positive entire solution with $u(x) \to l$ as $|x| \to \infty$.

As the first aim of this paper, we extend Naito's result to case (1.1). The formulation is as follows.

Assumption 1.2. There exist continuous functions κ, β on $(0, \infty)$ such that

$$|K(x)| \le \kappa(|x|), \qquad |f(x)| \le \beta(|x|), \qquad \int_0^\infty t\kappa(t)\mathrm{d}t =: A < \infty, \qquad \int_0^\infty t\beta(t)\mathrm{d}t =: F < \infty.$$

Remark 1.3. Assumption 1.2, in particular, includes case (1.2) with $l < -2$.

Theorem 1.4. *Under Assumptions 1.1 and 1.2, there exists a strictly positive constant μ^* depending only on n, p, A, F such that for any $\mu \in [0, \mu^*)$ we have an interval I_μ and for any $l \in I_\mu$ there exists a positive entire solution u of (1.1) satisfying $u(x) \to l$ as $|x| \to \infty$. In fact, we take*

$$0 < \mu_* := \sup\{\mu \in [0, \bar{\mu}) : a_\mu < b_\mu\}, \qquad \bar{\mu} = \left(\frac{n-2}{p}\right)^{\frac{p}{p-1}} \cdot A^{-\frac{1}{p-1}} \cdot F^{-1} \tag{1.5}$$

and for $\mu \in [0, \mu^*)$ we set $I_\mu = \left(a_\mu, b_\mu\right]$ by the smallest nonnegative root of $a_\mu = a(n, p, A, F, \mu)$ the equation of the unknown l, $l = (n-2)^{-1}(Al^p + \mu F)$ and $b_\mu := \left(1 - \frac{1}{p}\right)\left(\frac{n-2}{pA}\right)^{\frac{1}{p-1}} - \frac{\mu F}{n-2}$.

We are to check if μ_* and I_μ are well defined.

Remark 1.5. The quantity $\bar{\mu}$ is the supremum among μ for which b_μ is positive. We note that $a_0 = 0 < b_0 = \left(1 - \frac{1}{p}\right)\left(\frac{n-2}{pA}\right)^{\frac{1}{p-1}}$ when $\mu = 0$. Moreover, a_μ is continuous and strictly increasing as μ increases whereas b_μ is continuous and strictly decreasing. Hence, μ_* exists as a strictly positive constant. The intervals I_μ are set in a way that, as long as $\mu \in [0, \mu_*)$, for any $l \in I_\mu$ the following holds:

$$l > 0, \quad l - (n-2)^{-1}(Al^p + \mu F) > 0, \qquad l = c - (n-2)^{-1}(Ac^p + \mu F) \quad \text{for some } c > 0.$$

The second objective of this paper is to show that (1.1) in the radial case

$$u_{rr} + \frac{n-1}{r} u_r + k(r) u^p + \mu g(r) = 0, \quad u = u(r), \tag{1.6}$$

has *the structure of partial separation*, meaning that there exists an interval (α_μ, β_μ) with $0 \le \alpha_\mu < \beta_\mu \le \infty$ such that (1.6) possesses an entire solution u_ξ for each ξ in an interval (α_μ, β_μ) and any two of them do not intersect. The formulation is as follows.

Theorem 1.6. *Under Assumptions 1.1 and 1.2 in the radial case with $K(x) = k(|x|)$ and $f(x) = g(|x|)$, there exists μ_0 such that, for any $\mu \in [0, \mu_0)$, we have two intervals $I_{\mu,1}, I_{\mu,2}$ in $(0, \infty]$ such that for each $\alpha \in I_{\mu,1}$ Eq. (1.6) has a positive entire solution u_α with $u_\alpha(0) = \alpha$ which converges to $l \in I_{\mu,2}$ at ∞ and $u_{\alpha_1} < u_{\alpha_2}$ for any $\alpha_1 < \alpha_2$ in $I_{\mu,1}$. Moreover, the map α to l is one-to-one and onto from $I_{\mu,1}$ to $I_{\mu,2}$.*

In the proof of Theorem 1.6 the main ingredient is Proposition 6.1 in [4]; in fact, Theorem 1.6 is an extension of the proposition.

Third, we also consider the stability of solutions of the Cauchy problem:

$$\begin{cases} u_t = \Delta u + K(x)u^p + f(x) & \text{in } \mathbf{R}^n \times (0, \infty), \\ u(x, 0) = \phi(x) & \text{in } \mathbf{R}^n. \end{cases} \tag{1.7}$$