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The existence of asymptotically stable solutions for a nonlinear functional integral equation with values in a general Banach space

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ABSTRACT

Motivated by the recent known results about the solvability and existence of asymptotically stable solutions for nonlinear functional integral equations in spaces of functions defined on unbounded intervals with values in the n-dimensional real space, we establish asymptotically stable solutions for a nonlinear functional integral equation in the space of all continuous functions on $\mathbb{R}_+$ with values in a general Banach space, via a fixed point theorem of Krasnosel'skii type. In order to illustrate the result obtained here, an example is given.

1. Introduction

In this paper, we consider the solvability and the existence of asymptotically stable solutions for the following nonlinear functional integral equation:

$$x(t) = q(t) + f(t, x(t)) + \int_0^t V\left(t, s, x(s), \int_0^s V_1(t, s, r, x(r))\, dr\right) ds$$
$$+ \int_0^\infty G\left(t, s, x(s), \int_0^s G_1(t, s, r, x(r))\, dr\right) ds, \quad t \in \mathbb{R}_+, \tag{1.1}$$

where $q : \mathbb{R}_+ \to E; f : \mathbb{R}_+ \times E \to E; V : \Delta \times E^2 \to E; V_1 : \Delta_1 \times E \to E; G : \mathbb{R}_+^2 \times E^2 \to E; G_1 : \mathbb{R}_+ \times \Delta \times E \to E$ are supposed to be continuous and $\Delta = \{(t, s) \in \mathbb{R}_+^2 : s \le t\}$, $\Delta_1 = \{(t, s, r) \in \mathbb{R}_+^3 : r \le s \le t\}$ and E is a general Banach space.

Nonlinear functional integral equations with bounded intervals or unbounded intervals have been studied extensively by many authors using various methods and techniques. There are many important results about the existence, stability and other properties of solutions; for example, we refer the reader to [1–17] and the references given therein.

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Our results improve and generalize in part the results recently established in [1,2,6,11], for $E = \mathbb{R}^d$. In [1,2], Avramescu and Vladimirescu have proved the existence of asymptotically stable solutions to the following integral equations:

$$x(t) = q(t) + f(t, x(t)) + \int_0^t V(t, s)x(s)\mathrm{d}s + \int_0^t G(t, s, x(s))\mathrm{d}s, \quad t \in \mathbb{R}_+, \tag{1.2}$$

or

$$x(t) = q(t) + \int_0^t K(t, s, x(s))\mathrm{d}s + \int_0^\infty G(t, s, x(s))\mathrm{d}s, \quad t \in \mathbb{R}_+, \tag{1.3}$$

where the functions given with real values are supposed to be continuous, satisfying suitable conditions. In the proofs, the following fixed point theorem of Krasnosel'skii type is used (see [1,2]).

Also applying a fixed point theorem of Krasnosel'skii type and giving the suitable assumptions, Dhage and Ntouyas [6], Purnaras [11] obtained some results on the existence of solutions to the following nonlinear functional integral equation:

$$x(t) = q(t) + \int_0^{\mu(t)} k(t, s)f(s, x(\theta(s)))\mathrm{d}s + \int_0^{\sigma(t)} v(t, s)g(s, x(\eta(s)))\mathrm{d}s, \quad t \in [0, 1], \tag{1.4}$$

where $E = \mathbb{R}$, $0 \le \mu(t) \le t$; $0 \le \sigma(t) \le t$; $0 \le \theta(t) \le t$; $0 \le \eta(t) \le t$, for all $t \in [0, 1]$. Purnaras also shows that the technique used in [11] can be applied to yield existence results for the following equation:

$$\begin{aligned} x(t) = {} & q(t) + \int_{\alpha(t)}^{\mu(t)} k(t, s)f(s, x(\theta(s)))\mathrm{d}s \\ & + \int_{\beta(t)}^{\lambda(t)} \widehat{k}(t, s)F\left(s, x(\nu(s)), \int_0^{\sigma(s)} k_0(s, v, x(\eta(v)))\mathrm{d}v\right)\mathrm{d}s, \quad t \in [0, 1]. \end{aligned} \tag{1.5}$$

For the case where the Banach space E is arbitrary, in [9], the existence of asymptotically stable solutions of the integral equation

$$x(t) = q(t) + f(t, x(t), x(\pi(t))) + \int_0^t V(t, s, x(s), x(\sigma(s)))\mathrm{d}s + \int_0^t G(t, s, x(s), x(\chi(s)))\mathrm{d}s, \quad t \in \mathbb{R}_+, \tag{1.6}$$

was proved by using the fixed point theorem of Krasnosel'skii type that follows.

Theorem 1.1. *Let $(X, |\cdot|_n)$ be a Fréchet space and let $U, C : X \to X$ be two operators.*

Assume that:

(i) *U is a k-contraction operator, with $k \in [0, 1)$ (depending on n), with respect to a family of seminorms $\|\cdot\|_n$ equivalent to the family $|\cdot|_n$;*
(ii) *C is completely continuous;*
(iii) $\lim_{|x|_n \to \infty} \frac{|Cx|_n}{|x|_n} = 0, \ \forall n \in \mathbb{N}$.

Then $U + C$ has a fixed point.

Also applying Theorem 1.1 and adding some suitable conditions, we also get the same results for (1.1) as were obtained for (1.6) in [9]. On the other hand, the proof is obtained by combination of the arguments in [9], some techniques in [1,2] with appropriate modifications and, especially, the arguments of density. The paper consists of four sections, and the existence of solutions and the existence of asymptotically stable solutions for (1.1) will be presented in Sections 2 and 3. Finally, we give an illustrated example.

2. The existence of solutions

Let $X = C(\mathbb{R}_+; E)$ be the space of all continuous functions on $\mathbb{R}_+$ to E which are equipped with the denumerable family of seminorms

$$|x|_n = \sup_{t \in [0,n]} |x(t)|, \quad n \ge 1.$$

Then $(X, |\cdot|_n)$ is complete in the metric

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x - y|_n}{1 + |x - y|_n}$$

and X is the Fréchet space. Consider in X the other family of seminorms $\|\cdot\|_n$, defined as follows:

$$\|x\|_n = |x|_{\gamma_n} + |x|_{h_n}, \quad n \ge 1,$$