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A priori estimates, existence and non-existence of positive solutions of generalized mean curvature equations

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ABSTRACT

This paper concerns a priori estimates and existence of solutions of generalized mean curvature equations with Dirichlet boundary value conditions in smooth domains. Using the blow-up method with the Liouville-type theorem of the p laplacian equation, we obtain a priori bounds and the estimates of interior gradient for all solutions. The existence of positive solutions is derived by the topological method. We also consider the non-existence of solutions by Pohozaev identities.

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1. Introduction

We consider the generalized mean curvature equation

$$\begin{cases} -\mathrm{div}((1+|Du(x)|^2)^{\frac{p}{2}-1}Du(x)) = u^q(x), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where p, q are real positive numbers, Ω is a bounded domain with smooth boundary $\partial\Omega$, $\Omega \subset \mathbb{R}^N$, $1 < p < N$, $N \geq 2$ is the dimension of Ω. We call $-\mathrm{div}((1+|Du(x)|^2)^{\frac{p}{2}-1}Du(x))$ the generalized mean curvature operator. Let $p^* = \frac{Np}{N-p}$ be the critical exponent of the generalized mean curvature operator. We focus on the case of $p-1 < q < p^*-1$. Let us mention the importance of this operator. In the case of $p = 1$, it turns out to be the mean curvature operator, which has been attracting much attention in differential geometry and partial differential equation. The interested readers may refer to [1,2] and references therein for more details. In the case of $p = 2$, it becomes the classical Laplacian operator. It has been extensively studied from many aspects such as variational approaches, a priori estimates and so on. It also has many important applications in physics, biology and other interdisciplines. In this paper, we mainly consider the generalized mean curvature equation from the aspect of a priori estimates, which further implies the existence of positive solutions. Although (1.1) has a variational structure and the existence can be obtained by the maximum–minimum theorem, the solutions obtained from these methods usually have no a priori bounds. The a priori estimates give more information about the solutions. Furthermore, if the variational method is not applicable, i.e. the Euler–Lagrange equation does not exist, the

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question of the existence of solutions can be also dealt with a priori estimates and topological methods. In the literature, several approaches are widely applied to elliptic equations to derive a priori estimates. (1) The method of Rellich–Pohozaev identities together with moving planes in [3]. The solution around the boundary is estimated by the moving plane method and the interior bound is estimated by Rellich–Pohozaev identities and bootstrap method. (2) The rescaling or blow-up method and suitable Liouville-type theorem (see. e.g. [4]). The method is carried out by contradiction by assuming that there exists a sequence of unbounded solutions. Under the appropriate rescaling and elliptic estimates, the sequences converge to a solution in the whole space or halfspace, which contradicts the corresponding Liouville-type theorem. We applied this idea in our proofs. (3) The method of Hardy–Sobolev inequalities, which is introduced in [5]. The idea is to use the first eigenfunction of the Laplacian operator as multiples, Hardy–Sobolev inequalities and bootstrap arguments to derive a uniform bound.

We study the a priori estimates of positive solutions of (1.1) and prove the uniform bound on full sets of solutions. Our main results can be stated as follows.

Theorem 1. *Let Ω be bounded smooth domain and $p-1<q<p^*-1$. Assume that $u(x)\in W_0^{1,p}(\Omega)$ is any weak solution of (1.1), then there exists M independent of u such that $\|u\|_{L^\infty(\Omega)}\le M$.*

We also have the following corollary.

Corollary 1. *Let Ω be bounded smooth domain, the mean curvature of $\partial\Omega$ have positive bounds H_0 from below and $p-1< q<p^*-1$. Assume that $u\in W_0^{1,p}(\Omega)$ is a weak solution of (1.1), then u is a C^2 classical solution. Moreover, there exists a constant C independent of u such that $\|u\|_{C^2(\bar{\Omega})}\le C$.*

Based on a priori estimates, we obtain the existence of nonnegative solutions in (1.1) by the fixed point theorem.

Theorem 2. *Let Ω be bounded smooth domain, the mean curvature of $\partial\Omega$ have positive bounds H_0 from below, $p\ge 2$ and $p-1<q<p^*-1$. Then Eq. (1.1) has at least one positive solution.*

Remark 1. The above theorems hold for a more general function $f(u)$ instead of u^q. We only prove this case for simplicity.

Furthermore, we take into account the non-existence of the nonnegative solutions of (1.1). We establish the following theorem.

Theorem 3. *Suppose that Ω is a smooth bounded domain in $\mathbb{R}^N$, which is strictly star-shaped with respect to the origin in $\mathbb{R}^N$. Then, any solution $u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ in (1.1) vanishes identically in the case of $q\ge p^*-1$.*

We also show the comparison principle for the generalized mean curvature operator, which is included in a general form in [1]. We prove it in a different way that is very similar to the p laplacian equation for completeness. The comparison principle plays an important role in proving the boundary gradient estimates and the existence theorem.

This paper is organized as follows. In Section 2, we present some preliminary knowledge, which includes Liouville-type theorems for the p laplacian equation in Euclidean space and halfspace. Section 3 is devoting to establishing a priori estimates for every positive solution. We first obtain the a priori bound by the blow-up method and Liouville-type theorems. Then, we show the interior gradient estimate by the control of the distance function using a newly developed rescaling method. In Section 3, we derive the existence theorem of this type of quasi-linear equation following from the existence of fixed points on compact operators in a cone. The non-existence theorem is shown in Section 4. In Appendix, we give the proof of the comparison principle for the generalized mean curvature operator. Throughout the paper, C, C_0 and M denote generic positive constants, which is independent of u and may vary from line to line.

2. Preliminary knowledge

In this section, we collect some results for the quasi-linear elliptic equation and Liouville-type theorems for the p laplacian equation. The generalized mean curvature equation is a typical quasi-linear equation with principal part in divergence form. Let $a_i(Du)=(1+|Du|^2)^{\frac{p}{2}-1}D_iu$ and $p_i=D_iu$. The following structure condition holds:

$$\nu(p)(1+|Du|)^{p-2}\sum_{i=1}^N\xi_i^2\le\sum_{i,j=1}^N\frac{\partial_ia_i(Du)}{\partial p_j}\xi_i\xi_j\le\mu(p)(1+|Du|)^{p-2}\sum_{i=1}^N\xi_i^2, \tag{2.1}$$

where $\mu(p)$, $\nu(p)$ are positive constants with respect to p. If $\max_\Omega|Du|$ is bounded, the generalized mean curvature equation is strictly elliptic for $1<p<N$.

The next lemma is the Liouville-type theorem for the p laplacian equation in $\mathbb{R}^N$ [6].