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Global asymptotic stability for damped half-linear oscillators

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ABSTRACT

A necessary and sufficient condition is established for the equilibrium of the oscillator of half-linear type with a damping term,

$$\left(\phi_p(x')\right)' + h(t)\phi_p(x') + \phi_p(x) = 0$$

to be globally asymptotically stable. The obtained criterion is given by the form of a certain growth condition of the damping coefficient $h(t)$ and it can be applied to not only the cases of large damping and small damping but also the case of fluctuating damping. The presented result is new even in the linear cases ($p = 2$). It is also discussed whether a solution of the half-linear differential equation

$$\left(r(t)\phi_p(x')\right)' + c(t)\phi_p(x) = 0$$

that converges to a non-zero value exists or not. Some suitable examples are included to illustrate the results in the present paper.

1. Introduction

The purpose of this paper is to show that a growth condition on $h(t)$ is a necessary and sufficient condition for the equilibrium of the second-order differential equation

$$\left(\phi_p(x')\right)' + h(t)\phi_p(x') + \phi_p(x) = 0 \tag{HL}$$

to be globally asymptotically stable. Here, the prime denotes $\mathrm{d}/\mathrm{d}t$, the function $\phi_p(z)$ is defined by

$$\phi_p(z) = |z|^{p-2}z, \quad z \in \mathbb{R}$$

with $p > 1$, and the damping coefficient $h(t)$ is continuous and nonnegative for $t \ge 0$. Let

$$H(t) = \int_0^t h(s)\mathrm{d}s.$$

If $x(t)$ is a solution of (HL), then the function $cx(t)$ is another solution of (HL), where c is an arbitrary constant except 1. In general, however, the total of two solutions of (HL) is not a solution of (HL). Hence, the solution space of (HL) is homogeneous, but not additive. Because there is only characteristic half of the solution space of linear differential equations, Eq. (HL) is often called *half-linear*.

Let $\mathbf{x}(t) = (x(t), x'(t))$ and $\mathbf{x}_0 \in \mathbb{R}^2$, and let $\|\cdot\|$ be any suitable norm. We denote the solution of (HL) through $(t_0, \mathbf{x}_0)$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. It is clear that Eq. (HL) has the equilibrium $\mathbf{x}(t) \equiv \mathbf{0}$.

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The equilibrium is said to be *stable* if, for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The equilibrium is said to be *attractive* if, for any $t_0 \geq 0$, there exists a $\delta_0(t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta_0$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \to 0$ as $t \to \infty$. The equilibrium is said to be *globally attractive* if, for any $t_0 \geq 0$, $\eta > 0$ and $\mathbf{x}_0 \in \mathbb{R}^2$, there is a $T(t_0, \eta, \mathbf{x}_0) > 0$ such that $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T(t_0, \eta, \mathbf{x}_0)$. The equilibrium is *asymptotically stable* if it is stable and attractive. The equilibrium is *globally asymptotically stable* if it is stable and globally attractive. For the definitions, refer the books [1–9] for example.

Since $\phi_2(z) = z$, we can consider the damped linear oscillator

$$x'' + h(t)x' + x = 0 \tag{L}$$

as a special case of (HL). In the linear differential equations such as Eq. (L), it is well known that the equilibrium is attractive (resp., asymptotically stable), then it is globally attractive (resp., globally asymptotically stable). The study of the (global) asymptotic stability for Eq. (L) (or its general type) is one of the major themes in the qualitative theory of differential equations. Numerous papers have been devoted to find sufficient conditions and necessary conditions for the asymptotic stability (for example, see [10–21]).

We can cite Levin and Nohel [18, Theorem 1] as a pioneering work (their result can be applied to more general equations than Eq. (L)). They proved that if there exist two positive constants $\underline{h}$ and $\overline{h}$ such that $\underline{h} \leq h(t) \leq \overline{h}$ for $t \geq 0$, then the equilibrium of (L) is asymptotically stable. The research afterwards have advanced towards the direction where at least one of the lower bound $\underline{h}$ or the upper bound $\overline{h}$ is taken off. The case in which $\underline{h} \leq h(t) < \infty$ for $t \geq 0$ and the case in which $0 \leq h(t) \leq \overline{h}$ for $t \geq 0$ are often called *large damping* and *small damping*, respectively.

In the case of large damping, we should first make the special mention of Smith [21, Theorems 1 and 2]. He proved that

$$\int_0^\infty \frac{\int_0^t e^{H(s)}\mathrm{d}s}{e^{H(t)}}\,\mathrm{d}t = \infty \tag{1.1}$$

is a necessary and sufficient condition for the equilibrium of (L) to be asymptotically stable. Later, Ballieu and Peiffer [11] obtained several sufficient conditions and necessary conditions for the equilibrium of a certain kind of nonlinear differential equation to be globally asymptotically stable and presented the same criterion as Smith's by using their results (see [11, Corollary 6]). Although the expression of condition (1.1) is very concise, it is not so easy to confirm whether condition (1.1) is satisfied. For this reason, many attempts were carried out to look for other growth conditions that guarantee the asymptotic stability for Eq. (L) or more general nonlinear equations. Artstein and Infante [10] showed that if $H(t)/t^2$ is bounded for t sufficiently large, then the equilibrium of (L) is asymptotically stable. When an indefinite integral of $h(t)$ can be obtained, we can confirm their growth condition. In Artstein and Infante's result, the exponent 2 is best possible in the meaning that it cannot be changed to any $2+\varepsilon$, $\varepsilon > 0$. However, their growth condition is weaker than condition (1.1) because it is sufficient for the asymptotic stability, but not necessary. For example, consider Eq. (L) with large damping $h(t) = (2 + t)\log(2 + t)$. Then, it is easy to check that $H(t)/t^2$ is unbounded. Hence, Artstein and Infante's result is unavailable. However, it is known that the equilibrium of (L) with $h(t) = (2 + t)\log(2 + t)$ is asymptotically stable (see [11, Corollary 7]). Hatvani et al. [15] proved that the growth condition (1.1) on $h(t)$ is equivalent to

$$\sum_{n=1}^{\infty}\left(H^{-1}(nc) - H^{-1}((n-1)c)\right)^2 = \infty \tag{1.2}$$

for any $c > 0$, where $H^{-1}(s)$ denote the inverse function of $s = H(t)$. The merit of the discrete criterion (1.2) is that it is sometimes easier to check it. For example, we see that if $h(t) = t$, then condition (1.2) is satisfied; if $h(t) = t^2$, then condition (1.2) is not satisfied. However, in general case, it is still difficult to verify condition (1.2). Fortunately, unlike old times, there is a possibility that condition (1.1) can be confirmed by using numerical analysis conducted via personal computer even if it is impossible by the human hand calculation.

Ballieu and Peiffer [11, Theorems 5 and 6] also discussed the case of small damping. From their results, we see that the equilibrium of (L) is asymptotically stable if and only if $H(t)$ tends to ∞ as $t \to \infty$, provided that $h(t)$ is positive and nonincreasing for $t \geq 0$. Later, in the case of small damping, Hatvani [13, Corollary 4.4] showed that the weak integral positivity of $h(t)$ implies the asymptotic stability for Eq. (L) (see also [22,23]). For the definition of the weak integral positivity, see Section 3. Moreover, Hatvani [14, Theorem 1.1] proved that if $\limsup_{t\to\infty} H(t)/t^{2/3} > 0$, then the equilibrium of (L) is asymptotically stable and pointed out that the exponent $2/3$ is best possible in the meaning that it cannot be changed to any $2/3 - \varepsilon$, $\varepsilon > 0$. Although his condition is very sharp, it is not necessary and sufficient for the asymptotic stability. There are a lot of other works in the case of small damping, but no necessary and sufficient conditions such as (1.1) has been reported at all.

The case in which $h(t)$ has neither the lower bound $\underline{h}$ nor the upper bound $\overline{h}$ may be most difficult in the study of the asymptotic stability for Eq. (L). Let us call such a case *fluctuating damping*. Pucci and Serrin [19, Theorem A] considered N-dimensional nonlinear systems which contain Eq. (L) as a special case and presented sufficient conditions and necessary conditions for the global asymptotic stability applied even to the case of fluctuating damping (see also [20]). As another result that can even be applied to the case of fluctuating damping, we can cite Hatvani and Totik [16, Theorem 3.1]. We will compare our result with the result of them in the last part of this paper.