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Stability of linear neutral differential equations with delays and impulses established by the fixed points method

Dianli Zhao [a,b,*], Dong Han [a]

[a] Department of Mathematics, Shanghai Jiaotong University, Shanghai, 200240, China

[b] College of Science, University of Shanghai for Science and Technology, Shanghai, 200093, China

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ABSTRACT

In this paper, we consider the impulsive effects on the stability of the zero solution of the linear neutral differential equations with variable delays. By transforming the equations into ones without impulses and using fixed point theory, some sufficient conditions for asymptotic stability and exponential stability of the zero solution are obtained. The paper extends and improves results on sufficient conditions obtained by Jin and Luo (2008) [17], and Ardjouni and Djoudi (2011) [18], which is shown clearly in Example 1. This paper also shows that the impulse intensity and the impulse time both influence the decay rate of the convergence to zero of the solutions. Finally, two examples are given to show applications of some results obtained.

1. Introduction

The purpose of this paper is to study the stability of the zero solution of the first-order linear neutral differential equations with variable delays and impulses

$$\begin{cases} x'(t) = -b(t)\,x(t-\tau(t)) + c(t)\,x'(t-\tau(t)), & t \neq t_k \\ x\left(t_k^+\right) - x(t_k) = d_k x(t_k), & k = 1, 2, \ldots \end{cases} \tag{1}$$

and the generalized form

$$\begin{cases} x'(t) = -a(t)\,x(t) - \displaystyle\sum_{j=1}^{N} b_j(t)\,x\left(t-\tau_j(t)\right) + \sum_{j=1}^{N} c_j(t)\,x'\left(t-\tau_j(t)\right), & t \neq t_k \\ x\left(t_k^+\right) - x(t_k) = d_k x(t_k), \quad k = 1, 2, \ldots \end{cases} \tag{2}$$

by a fixed point method under the following assumptions:

(H1) $0 \leq \sigma < t_1 < t_2 \cdots < t_k < \cdots$ are fixed points with $t_k \to \infty$ as $k \to \infty$.
(H2) $a, b, c, b_j, c_j \in C\left(R^+, R\right)$, $\tau, \tau_j \in C\left(R^+, R^+\right)$, $t - \tau(t) \to \infty$ and $t - \tau_j(t) \to \infty$ as $t \to \infty$.
(H3) N is a positive integer and $d_k \in (-1, \infty)$ are constants for $k = 1, 2, \ldots$.
(H4) $\lim_{t \to t_k^-} x(t) = x\left(t_k^-\right)$ and $\lim_{t \to t_k^+} x(t) = x\left(t_k^+\right)$ for $k = 1, 2, \ldots$.

Study on the stability of linear neutral differential equations is not a new topic. But there are a large set of problems for which it has been ineffective when Lyapunov's direct method has been used. Recently, Xing and Han [1] made the conditions

* Corresponding author at: Department of Mathematics, Shanghai Jiaotong University, Shanghai, 200240, China. Tel.: +86 21 65691507.
E-mail address: Tc_zhaodianli@139.com (D. Zhao).

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ensuring stability simpler and less restrictive by constructing several functions of partial components of x instead of putting all components of the state variable x in one Lyapunov function. On the other hand, Burton and others [2–10] applied the fixed point theory to study stability. It has been shown that many of those problems encountered in the study of stability by means of Lyapunov's direct method can be solved by means of the fixed point theory. After that, Luo [11] first considered the exponential stability for stochastic partial differential equations with delays by using a fixed point method. Then, together with Luo, Sakthivel [12,13] investigated the asymptotic stability of the nonlinear impulsive stochastic differential equations and the impulsive stochastic partial differential equations with infinite delays by means of fixed point theory. By using the fixed point theory to study the stability of the zero solutions of linear neutral differential equations and the special cases without impulses, many results have been shown in [14–16] and the references therein. For example, Jin and Luo in [17] have studied the equation

$$x'(t) = -a(t)x(t) - b(t)x(t-\tau(t)) + c(t)x'(t-\tau(t)) \tag{3}$$

and obtained the following result.

Theorem 1.1 ([17]). *Let $\tau(t)$ be twice differentiable and suppose that $\tau'(t) \neq 1$ for all $t \in R$. Suppose that there exist a constant $0 < \alpha < 1$ and a continuous function $h : R^+ \to R$ such that for $t \geq 0$,*

$$\liminf_{t\to\infty} \int_0^t h(s)\,\mathrm{d}s > -\infty,$$

and

$$\left|\frac{c(t)}{1-\tau'(t)}\right| + \int_{t-\tau(t)}^t |h(s)-a(s)|\,\mathrm{d}s + \int_0^t \mathrm{e}^{-\int_s^t h(u)\mathrm{d}u} \Big| -b(s) + [h(s-\tau(s)) - a(s-\tau(s))](1-\tau'(s)) - r(s)\Big|\,\mathrm{d}s + \int_0^t \mathrm{e}^{-\int_s^t h(u)\mathrm{d}u} |h(s)| \left(\int_{s-\tau(s)}^s |h(u)-a(u)|\,\mathrm{d}u\right) \mathrm{d}s \leq \alpha, \tag{4}$$

where $r(s) = \frac{(h(s)c(s)+c'(s))(1-\tau'(s))+c(s)\tau''(s)}{(1-\tau'(s))^2}$.

Then the zero solution of (3) is asymptotically stable if and only if $\int_0^t h(u)\,\mathrm{d}u \to \infty$ as $t \to \infty$.

When $a(t) \equiv c(t) \equiv 0$, Theorem 1.1 with $h(s) \equiv b(g(s))$ reduces to Theorem 2.2 in [15]. On choosing $h(s) \equiv a(s)$, Theorem 1.1 reduces to Theorem 2.1 in [16]. The stability of the generalized linear neutral differential equations with variable delays has been investigated in [18]. All the above papers do not consider the effect of impulses. However, in reality, many phenomena in the fields of medicine, biology, economics and chemistry do exhibit impulse effects. The monographs [19,20] are good sources for the study of impulsive differential equations and their applications. The recent survey papers [21,22] provide the oscillation theory of impulsive ordinary differential equations and impulsive delay differential equations. To the best of our knowledge, there is no result on the stability of the linear neutral differential equations with variable delays and impulses of type (1) or (5) presented by using the fixed point theory.

This paper is organized as follows. Section 2 includes notation, definitions and some lemmas needed in the later sections. In Section 3, the linear impulsive delay differential equations are discussed and sufficient conditions for stability are presented. In the last part, Section 4, we deal with two examples.

2. Preliminary notes

Suppose that $R = (-\infty, +\infty)$, $R^+ = [0, +\infty)$ and $Z^+ = \{1, 2, 3, \ldots\}$. $C(S_1, S_2)$ denotes the set of all continuous functions $\varphi : S_1 \to S_2$. For each $\sigma \in R^+$, define $m(\sigma) = \inf\{s-\tau(s) : s \geq \sigma\}$, $m_j(\sigma) = \inf\{s-\tau_j(s) : s \geq \sigma\}$ and $C(\sigma) = C([m(\sigma), \sigma], R)$ with the supremum norm $\|\psi\| = \max\{|\psi(s)| : m(\sigma) \leq s \leq \sigma\}$. Define $n \triangleq n(t) = \max\left\{k \in Z^+ : t_k < t\right\}$. For simplicity, we define $\prod_{u\leq t_k<v}(\cdot) \triangleq \prod_{k\in\{k|k\in Z^+ \text{ and } u\leq t_k<v\}}(\cdot)$ for all u, v $\in$ R. Here and in the sequel, we assume that a product equals unity if the number of factors is equal to zero.

Definition 2.1. For any $\sigma \geq 0$ and $\psi \in C(\sigma)$, a function $x : [m(\sigma), \infty) \to R$ denoted by $x(t, \sigma, \psi)$ is said to be a solution of (1) on $[\sigma, \infty)$ satisfying the initial value condition $x(t) = \psi(t)$ for $t \in [m(\sigma), \sigma]$, if the following conditions are satisfied:

(i) $x(t)$ is absolutely continuous on $[\sigma, t_1)$ and each interval (t_k, t_{k+1}).
(ii) $x\left(t_k^-\right)$ and $x\left(t_k^+\right)$ exist and $x\left(t_k^-\right) = x(t_k)$ for any $t_k \in [\sigma, \infty)$.
(iii) $x(t)$ satisfies (1) almost everywhere in $[\sigma, \infty)$, and may have a discontinuity of the first kind at t_k for $k = 1, 2, \ldots$.

Definition 2.2. For any $\psi \in C(\sigma)$, the zero solution of (1) is said to be:

(i) stable if for any $\varepsilon > 0$ and $\sigma \geq 0$, there exists a $\delta = \delta(\varepsilon, \sigma) > 0$ such that $\psi \in C(\sigma)$ and $\|\psi\| < \delta$ imply $|x(t, \sigma, \psi)| < \varepsilon$ for $t \geq \sigma$;
(ii) uniformly stable if δ is independent of σ;