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Existence of positive pseudo almost periodic solutions to a class of neutral integral equations ☆

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ABSTRACT

In this work, a new fixed point theorem in partially ordered Banach spaces is established, and then used to prove the existence of positive pseudo almost periodic solutions to a class of neutral integral equations. Our existence theorem extends some recent results due to [E. Ait Dads, P. Cieutat, L. Lhachimi, Positive pseudo almost periodic solutions for some nonlinear infinite delay integral equations, Mathematical and Computer Modelling 49 (2009) 721–739] to a more general class of integral equations.

1. Introduction

In [1], Fink and Gatica initiated the study on the existence of positive almost periodic solution to the following delay integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\mathrm{d}s, \tag{1.1}$$

which is a model for the spread of some infectious disease (cf. [2,3]). Since then, the existence of positive almost periodic type solution to Eq. (1.1) and its variants is extensively studied by many authors (see, e.g., [4–13] and references therein). Especially, in 1997, Ait Dads and Ezzinbi [4] considered the existence of positive almost periodic solutions for the following neutral integral equation

$$x(t) = \gamma x(t-\tau) + (1-\gamma)\int_{t-\tau}^{t} f(s, x(s))\mathrm{d}s; \tag{1.2}$$

in 2000, Ait Dads and Ezzinbi [5] investigated the existence of positive pseudo almost periodic solution to the following infinite delay integral equation

$$x(t) = \int_{-\infty}^{t} a(t-s)f(s, x(s))\mathrm{d}s; \tag{1.3}$$

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and, recently, Ait Dads et al. [7] discussed the existence of positive pseudo almost periodic solution to the following more general infinite delay integral equation

$$x(t) = \int_{-\infty}^{t} a(t, t-s) f(s, x(s)) \mathrm{d}s, \tag{1.4}$$

which unifies Eqs. (1.1) and (1.3).

The aim of this paper is to extend the results of [7] to the following more general integral equation

$$x(t) = \alpha(t)x(t-\beta) + \int_{-\infty}^{t} a(t, t-s) f(s, x(s)) \mathrm{d}s + h(t, x(t)), \tag{1.5}$$

where α, β, a, f, h satisfy some conditions recalled in Section 3. It is worth noting that Eqs. (1.1)–(1.4) are both special cases of Eq. (1.5). In addition, for the case of $h = 0$, the existence of almost automorphic solution to Eq. (1.5) was studied in [8]. However, the approach used in [8] is different from the one we adopt here. In our paper, we establish a new fixed point theorem in partially ordered Banach spaces, and then apply the fixed point theorem to solve Eq. (1.5).

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the set of nonnegative real numbers, by X a real Banach space with the norm $\|\cdot\|$, by Ω a subset of X, and by $BC(\mathbb{R}, X)$ the set of all X-valued bounded continuous functions. First, let us recall some definitions, notations and basic results of almost periodicity and pseudo almost periodicity. For more details, we refer the reader to [14–16].

Definition 2.1. A continuous function $f: \mathbb{R} \to X$ is called almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t+\tau) - f(t)\| < \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

We denote by $AP(X)$ the set of all such functions.

Definition 2.2. A continuous function $f: \mathbb{R} \times \Omega \to X$ is called almost periodic in t uniformly for $x \in \Omega$ if for each $\varepsilon > 0$ and for each compact subset $K \subset \Omega$ there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t+\tau, x) - f(t, x)\| < \varepsilon \quad \text{for all } t \in \mathbb{R},\ x \in K.$$

We denote by $AP(\mathbb{R} \times \Omega, X)$ the set of all such functions.

Let

$$PAP_0(X) = \left\{ f \in BC(\mathbb{R}, X) : \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \|f(t)\| \mathrm{d}t = 0 \right\}.$$

Similarly, $PAP_0(\mathbb{R} \times X, X)$ denotes the collection of all jointly continuous functions $f: \mathbb{R} \times X \to X$ such that for each compact subset $K \subset \Omega$, f is bounded on $\mathbb{R} \times K$, and

$$\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \|f(t, x)\| \mathrm{d}t = 0 \quad \text{uniformly for } x \in K.$$

Definition 2.3. A function $f: \mathbb{R} \to X$ (respectively $f: \mathbb{R} \times \Omega \to X$) is called pseudo almost periodic (respectively pseudo almost periodic in t uniformly with respect to $x \in \Omega$) if

$$f = g + \phi,$$

where $g \in AP(X)$ (respectively $g \in AP(\mathbb{R} \times \Omega, X)$) and $\phi \in PAP_0(X)$ (respectively $\phi \in PAP_0(\mathbb{R} \times \Omega, X)$). We denote by $PAP(X)$ (respectively $PAP(\mathbb{R} \times \Omega, X)$) the set of all such functions.

Recall that pseudo almost periodic functions were introduced by Zhang in [17–19]. Recently, pseudo almost periodic functions and its generalizations have been of great interest for many mathematicians (see, e.g., [20–24,5,7] and references therein).

In the next section, we will need the following properties about pseudo almost periodic functions.

Lemma 2.4 (*[16]*). *The following hold true:*

(a) *$f, g \in PAP(X)$ implies that $f + g \in PAP(X)$.*
(b) *$f, g \in PAP(\mathbb{R})$ implies that $f \cdot g \in PAP(\mathbb{R})$.*
(c) *$f \in PAP(X)$ implies that $f(\cdot - c) \in PAP(X)$ for any constant $c \in \mathbb{R}$.*
(d) *$PAP(X)$ is a Banach space equipped with the supremum norm.*