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Higher-order radial derivatives and optimality conditions in nonsmooth vector optimization

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ABSTRACT

We propose notions of higher-order outer and inner radial derivatives of set-valued maps and obtain main calculus rules. Some direct applications of these rules in proving optimality conditions for particular optimization problems are provided. Then we establish higher-order optimality necessary conditions and sufficient ones for a general set-valued vector optimization problem with inequality constraints. A number of examples illustrate both the calculus rules and the optimality conditions. In particular, they explain some advantages of our results over earlier existing ones and why we need higher-order radial derivatives.

1. Introduction and preliminaries

In nonsmooth optimization, a large number of generalized derivatives have been introduced to replace the classical Fréchet and Gateaux derivatives to meet the continually increasing diversity of practical problems. We can recognize roughly two approaches: primal space and dual space approaches. Coderivatives, limiting subdifferentials and other notions in the dual space approach enjoy rich and fruitful calculus rules and little depend on convexity assumptions in applications; see e.g. the excellent book [1,2]. The primal space approach has been more developed so far, partially since it is more natural and exhibits clear geometrical interpretations. Most generalized derivatives in this approach are based on linear approximations and kinds of tangency. Hence, approximating cones play crucial roles. One of the early and most important notions are the contingent cone and the corresponding contingent derivative; see [3,4]. For a subset A of a normed space X, the contingent cone of A at $\bar{x} \in \operatorname{cl} A$ is

$$T_A(\bar{x}) = \{u \in X : \exists t_n \to 0^+, \exists u_n \to u, \forall n, \bar{x} + t_n u_n \in A\}.$$

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However, they capture only the local nature of sets and mappings and are suitable mainly for convex problems. The (closed) radial cone of A at $\bar{x} \in \text{cl}\, A$ is defined by

$$R_A(\bar{x}) = \overline{\text{cone}}(A - \bar{x}) = \{u \in X : \exists t_n > 0, \exists u_n \to u, \forall n, \bar{x} + t_n u_n \in A\}$$

and carries global information about A. We have $T_A(\bar{x}) \subseteq R_A(\bar{x})$ and this becomes equality if A is convex (in fact, we need A being only star-shape at $\bar{x}$). Hence, the corresponding radial derivative, first proposed in [5], is proved to be applicable to nonconvex problems and global optimal solutions. In [6,7], radial epiderivatives were introduced, taking some advantages of other kinds of epiderivatives; see e.g. [4,8]. A modified definition was included in [9], making the radial epiderivative a notion exactly corresponding to the contingent epiderivative defined in [4,8], to avoid some restrictive assumptions imposed in [6,7]. Radial epiderivatives were applied in [10] to get optimality conditions for strict minimizers.

To obtain more information for optimal solutions, higher-order (generalized) derivatives and higher-order optimality conditions have been intensively developed recently; see e.g. [11–15]. However, such contributions are still much fewer than the first and second-order considerations. Of course, only a number of generalized derivatives may have higher-order generalizations. As far as we know, the radial derivative has not had higher-order extensions so far. This is a motivation for our present work.

To meet various practical situations, many optimality (often known also as efficiency) notions have been introduced and developed in vector optimization. Each above-mentioned paper dealt with only several kinds of optimality. There were also attempts to classify solution notions in vector optimization. The Q-minimality proposed in [16] subsumes various types of efficiency, from weak and ideal solutions to many properly efficient solutions. Hence, when applying higher-order radial derivatives to establish optimality conditions, we start with Q-minimal solutions and then derive results for many other kinds of efficiency.

The layout of the paper is as follows. In the rest of this section, we recall some definitions and preliminaries for our later use. Section 2 includes definitions of higher-order outer and inner radial derivatives of set-valued mappings and their main calculus rules. Some illustrative direct applications of these rules for obtaining optimality conditions in particular problems are provided by the end of this section. The last section is devoted for establishing higher-order optimality conditions, in terms of radial derivatives, in a general set-valued vector optimization problem.

In the sequel, let X, Y and Z be normed spaces, $C \subseteq Y$ and $D \subseteq Z$ be pointed closed convex cones with nonempty interior. B_X, B_Y stands for the closed unit ball in X, Y, respectively. For $A \subseteq X$, $\text{int}A$, $\text{cl}\, A$, $\text{bd}A$ denote its interior, closure and boundary, respectively. Furthermore, $\text{cone}A = \{\lambda a \mid \lambda \geq 0,\ a \in A\}$. For a cone $C \subseteq Y$, we define:

$$C^* = \{y^* \in Y^* \mid \langle y^*, c\rangle \geq 0, \forall c \in C\},$$
$$C^{*i} = \{y^* \in Y^* \mid \langle y^*, c\rangle > 0, \forall c \in C \setminus \{0\}\}$$

and, for $u \in X$, $C(u) = \text{cone}(C + u)$. A convex set $B \subset Y$ is called a base for C if $0 \notin \text{cl}B$ and $C = \{tb : t \in \mathbb{R}+, b \in B\}$. For $H : X \to 2^Y$, the domain, graph and epigraph of H are defined by

$$\text{dom}\, H = \{x \in X : H(x) \neq \emptyset\}, \qquad \text{gr}\, H = \{(x, y) \in X \times Y : y \in H(x)\},$$
$$\text{epi}\, H = \{(x, y) \in X \times Y : y \in H(x) + C\}.$$

Throughout the rest of this section, let A be a nonempty subset of Y and $a_0 \in A$. The main concept in vector optimization is Pareto efficiency. Recall that a_0 is a Pareto minimal point of A with respect to (w.r.t.) C ($a_0 \in \text{Min}(A, C)$) if

$$(A - a_0) \cap (-C \setminus \{0\}) = \emptyset.$$

In this paper, we are concerned also with the following other concepts of efficiency.

Definition 1.1. (i) a_0 is a strong (or ideal) efficient point of A ($a_0 \in \text{StrMin}(A, C)$) if $A - a_0 \subseteq C$.

(ii) Supposing that $\text{int}\, C \neq \emptyset$, a_0 is a weak efficient point of A ($a \in \text{WMin}(A, C)$) if $(A - a_0) \cap (-\text{int}\, C) = \emptyset$.

(iii) Supposing that $C^{+i} \neq \emptyset$, a_0 is a positive-properly efficient point of A ($a_0 \in \text{Pos}(A, C)$) if there exists $\varphi \in C^{+i}$ such that $\varphi(a) \geq \varphi(a_0)$ for all $a \in A$.

(iv) a_0 is a Geoffrion-properly efficient point of A ($a \in \text{Ge}(A, C)$) if $a_0 \in \text{Min}(A, C)$ and there exists a constant $M > 0$ such that, whenever there is $\lambda \in C^+$ with norm one and $\lambda(a - a_0) > 0$ for some $a \in A$, one can find $\mu \in C^+$ with norm one such that

$$\langle \lambda, a - a_0\rangle \leq M\langle \mu, a_0 - a\rangle.$$

(v) a_0 is a Borwein-properly efficient point of A ($a \in \text{Bo}(A, C)$) if

$$\text{clcone}(A - a_0) \cap (-C) = \{0\}.$$

(vi) a_0 is a Henig-properly efficient point of A ($a_0 \in \text{He}(A, C)$) if there exists a convex cone K with $C \setminus \{0\} \subseteq \text{int}K$ such that $(A - a_0) \cap (-\text{int}K) = \emptyset$.

(vii) Supposing that C has a base B, a_0 is a strong Henig-properly efficient point of A ($a_0 \in \text{StrHe}(A, B)$) if there is a scalar $\epsilon > 0$ such that

$$\text{clcone}(A - a_0) \cap (-\text{clcone}(B + \epsilon B_Y)) = \{0\}.$$