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Extension of Caristi's fixed point theorem to vector valued metric spaces

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ABSTRACT

The paper deals with the classical Caristi fixed point theorem in vector valued metric spaces. The results obtained seem to be new in this setting.
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1. Introduction

In recent years, Caristi's fixed point theorem [1–3] has been the subject of intensive research. Recall that this theorem states that any map $T : M \to M$ has a fixed point provided that M is complete and there exists a lower semi-continuous map ϕ mapping M into the nonnegative numbers such that

$$d(x, Tx) \leq \phi(x) - \phi(Tx)$$

for every $x \in M$. This general fixed point theorem has found many applications in nonlinear analysis. It is shown, for example, that this theorem yields essentially all the known inwardness results [4] of geometric fixed point theory in Banach spaces. Recall that inwardness conditions are the ones which assert that, in some sense, points from the domain are mapped toward the domain. Possibly the weakest of the inwardness conditions, the Leray–Schauder boundary condition is the assumption that a map points x of ∂M anywhere except to the outward part of the ray originating at some interior point of M and passing through x.
The proofs given for Caristi's result vary and use different techniques (see [1,5,2,6]). It is worth mentioning that because of Caristi's result's close connection to Ekeland's [7] variational principle, many authors refer to it as the Caristi–Ekeland fixed point result. For more on Ekeland's variational principle and the equivalence between the Caristi–Ekeland fixed point result and the completeness of metric spaces, the reader is advised to read [8].

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In this work we prove a vector version of this theorem in vector valued metric spaces. The approach was used intensively in [9] where a Banach Contraction Principle was proved in this setting and used to obtain a result more general than the one obtained in [10] concerning isolated solutions of multi-point boundary value problems.

2. Caristi's fixed point theorem

The main motivation behind this work is the main point raised in [9] that the domain of existence of a solution to a system of first-order differential equations may be increased by considering vector valued distances. The example therein strengthens this point. In fact this point was noted by Bernfeld and Lakshmikantham [11] who indicated that "there is more flexibility working with generalized spaces" (meaning vector valued metric spaces).

Let $(\mathcal{V}, \preceq)$ be an ordered Banach space. The cone $\mathcal{V}_+ = \{v \in \mathcal{V}; \theta \preceq v\}$, where θ is the zero-vector of $\mathcal{V}$, satisfies the usual properties:

(1) $\mathcal{V}_+ \cap -\mathcal{V}_+ = \{\theta\}$,
(2) $\mathcal{V}_+ + \mathcal{V}_+ \subset \mathcal{V}_+$,
(3) $\alpha\, \mathcal{V}_+ \subset \mathcal{V}_+$ for all $\alpha \geq 0$.

The concept of vector valued metric spaces relies on the following definition.

Definition 1. Let M be a set. A map $d : M \times M \to \mathcal{V}$ defines a distance if:

(i) $d(x, y) = \theta$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for any $x, y \in M$,
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for any $x, y, z \in M$.

The pair (M, d) is called a vector valued metric space (vvms for short).

In [9] the following theorem, seen as a generalization of the Banach Contraction Principle, is proved.

Theorem 1. *Let (M, d) be a complete vvms, where $\mathcal{V} = \mathbb{R}^N$. Let $T : M \to M$. Assume there exists a positive operator $A : \mathbb{R}^N \to \mathbb{R}^N$, i.e. $A(\mathbb{R}^N_+) \subset \mathbb{R}^N_+$ which satisfies $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of A, such that*

$$d(T(x), T(y)) \preceq A\, d(x, y),$$

for any $x, y \in M$. Then:

(1) *there exists $\omega \in M$ such that for any $x_0 \in M$, the orbit $\{T^n(x_0)\}$ converges to ω, and moreover we have*

$$d(T^n(x_0), \omega) \preceq A^n (I - A)^{-1} d(x_0, T(x_0)) = \left(\sum_{k=n}^{\infty} A^k\right) d(x_0, T(x_0)),$$

for any $n \geq 1$;
(2) *the point ω is the only fixed point of T in M.*

As Caristi did for the case of the classical Banach Contraction Principle, let us discuss his ideas for the vvms. Under the assumptions of Theorem 1, where $\mathcal{V}$ is no longer the finite dimensional space $\mathbb{R}^N$, we have

$$d(T(x), T^2(x)) \preceq Ad(x, T(x))$$

for any $x \in M$, which implies

$$d(x, T(x)) + d(T(x), T^2(x)) \preceq d(x, T(x)) + Ad(x, T(x)).$$

Hence

$$d(x, T(x)) - Ad(x, T(x)) \preceq d(x, T(x)) - d(T(x), T^2(x))$$

or

$$(I - A)d(x, T(x)) \preceq d(x, T(x)) - d(T(x), T^2(x)).$$

Set $d_A(x, y) = (I - A)d(x, y)$, for any $x, y \in M$. Then it is easy to check that if $I - A$ is a positive one-to-one operator, then d_A is a vector valued distance defined on M. So if we set $F(x) = d(x, T(x))$, then we have

$$d_A(x, T(x)) \preceq F(x) - F(T(x)).$$

As Caristi did, one may wonder under what assumptions on any vvms (M, d) and $F : M \to \mathcal{V}_+$, any map $T : M \to M$ which satisfies

$$d(x, T(x)) \preceq F(x) - F(T(x)),$$

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