Nonlinear Analysis 74 (2011) 161–170

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Sharp Nash inequalities on manifolds with boundary in the presence of symmetries

Athanase Cotsiolis *, Nikos Labropoulos

Department of Mathematics, University of Patras, Patras 26110, Greece

ARTICLE INFO

Article history:
Received 10 December 2009
Accepted 19 August 2010

Keywords:
Manifolds with boundary
Symmetries
Trace Nash inequalities
Best constants

ABSTRACT

In this paper we establish the best constant $\widetilde{A}_{opt}(\overline{M})$ for the trace Nash inequality on a n-dimensional compact Riemannian manifold in the presence of symmetries, which is an improvement over the classical case due to the symmetries which arise and reflect the geometry of manifold. This is particularly true when the data of the problem is invariant under the action of an arbitrary compact subgroup G of the isometry group $Is(M, g)$, where all the orbits have infinite cardinal.

1. Introduction

We say that the Nash inequality (1) is valid if there exists a constant $A > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$

$$\left(\int_{\mathbb{R}^n} u^2 \mathrm{d}x\right)^{1+\frac{2}{n}} \leqslant A \int_{\mathbb{R}^n} |\nabla u|^2 \,\mathrm{d}x \left(\int_{\mathbb{R}^n} |u| \,\mathrm{d}x\right)^{\frac{4}{n}}. \tag{1}$$

Such an inequality first appeared in the celebrated paper of Nash [1], where he discussed the Hölder regularity of solutions of divergence form in uniformly elliptic equations. It is a particular case of the Gagliardo–Nirenberg type inequalities $\|u\|_r \leqslant C\|\nabla u\|_q^a \|u\|_s^{1-a}$ and it is well known that the Nash inequality (1) and the Euclidean type Sobolev inequality are equivalent in the sense that if one of them is valid, the other one is also valid (i.e. see [2]). It is, also, well known that with this procedure of passing from the one type of inequalities to the other, is impossible to compare the best constants, since the inequalities under use are not optimal.

As far as the optimal version of Nash inequality (1) is concerned, the best constant $A_0(n)$, that is

$$A_0(n)^{-1} = \inf\left\{ \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x \left(\int_{\mathbb{R}^n} |u| \mathrm{d}x\right)^{\frac{4}{n}}}{\left(\int_{\mathbb{R}^n} u^2 \mathrm{d}x\right)^{1+\frac{2}{n}}} \,\middle|\, u \in C_0^\infty(\mathbb{R}^n), u \not\equiv 0 \right\},$$

has been computed by Carlen and Loss in [3], together with the characterization of the extremals for the corresponding optimal inequality, as

$$A_0(n) = \frac{(n+2)^{\frac{n+2}{n}}}{2^{\frac{2}{n}} n \lambda_1^N |\mathcal{B}^n|^{\frac{2}{n}}},$$

* Corresponding author.
E-mail addresses: cotsioli@math.upatras.gr (A. Cotsiolis), nal@upatras.gr (N. Labropoulos).

0362-546X/$ – see front matter
doi:10.1016/j.na.2010.08.030

where $|\mathcal{B}^n|$ denotes the Euclidean volume of the unit ball $\mathcal{B}^n$ in $\mathbb{R}^n$ and λ_1^N is the first Neumann eigenvalue for the Laplacian for radial functions in the unit ball $\mathcal{B}^n$.

For an example of application of the Nash inequality with the best constant, we refer to Kato [4] and for a geometric proof with an asymptotically sharp constant, we refer to Beckner [5].

For compact Riemannian manifolds, the Nash inequality still holds with an additional L^1-term and that is why we will refer this as the L^1-Nash inequality.

Given (M, g) a smooth compact Riemannian n-manifold, $n \geq 2$, we get here the existence of real constants A and B such that for any $u \in C^\infty(M)$,

$$\left(\int_M u^2 \mathrm{d}V_g\right)^{1+\frac{2}{n}} \leqslant A \int_M |\nabla u|_g^2 \,\mathrm{d}V_g \left(\int_M |u| \mathrm{d}V_g\right)^{\frac{4}{n}} + B \left(\int_M |u| \mathrm{d}V_g\right)^{2+\frac{4}{n}}. \tag{2}$$

The best constant for this inequality is defined as

$$A_{\mathrm{opt}}^1(M) = \inf\left\{A > 0 : \exists B > 0 \text{ s.t. (2) is true } \forall u \in C^\infty(M)\right\}.$$

This inequality has been studied completely by Druet, Hebey and Vaugon. They proved in [6] that $A_{\mathrm{opt}}^1(M) = A_0(n)$, and (2) with its optimal constant $A = A_0(n)$ is sometimes valid and sometimes not, depending on the geometry of M.

Humbert in [7] studied the following L^2-Nash inequality

$$\left(\int_M u^2 \mathrm{d}V_g\right)^{1+\frac{2}{n}} \leqslant \left(A \int_M |\nabla u|_g^2 \,\mathrm{d}V_g + B \int_M u^2 \mathrm{d}V_g\right) \left(\int_M |u| \mathrm{d}V_g\right)^{\frac{4}{n}}, \tag{3}$$

for all $u \in C^\infty(M)$, of which the best constant is defined as

$$A_{\mathrm{opt}}^2(M) = \inf\left\{A > 0 : \exists B > 0 \text{ s.t. (3) is true } \forall u \in C^\infty(M)\right\}.$$

Contrary to the sharp L^1-Nash inequality, in this case, he proved that B always exists and $A_{\mathrm{opt}}^2(M) = A_0(n)$.

We denote $\mathbb{R}_+^n = \mathbb{R}^{n-1} \times [0, +\infty)$ and $\partial\mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}$. The trace Nash inequality states that a constant $\tilde{A} > 0$ exists such that for all $u \in C_0^\infty(\mathbb{R}_+^n)$, $n \geq 2$ with $\nabla u \in L^2(\mathbb{R}^n)$ and $u|_{\partial\mathbb{R}_+^n} \in L^1(\partial\mathbb{R}_+^n) \cap L^2(\partial\mathbb{R}_+^n)$

$$\left(\int_{\partial\mathbb{R}_+^n} u^2 \mathrm{d}s\right)^{\frac{n}{n-1}} \leqslant \widetilde{A} \int_{\mathbb{R}_+^n} |\nabla u|^2 \,\mathrm{d}x \left(\int_{\partial\mathbb{R}_+^n} |u| \mathrm{d}s\right)^{\frac{2}{n-1}}, \tag{4}$$

where $\mathrm{d}s$ is the standard volume element on $\mathbb{R}^{n-1}$ and the trace of u on $\partial\mathbb{R}_+^n$ is also denoted by u.

Let $\widetilde{A}_0(n)$ be the best constant in Nash inequality (4). That is

$$\widetilde{A}_0(n)^{-1} = \inf\left\{ \frac{\int_{\mathbb{R}_+^n} |\nabla u|^2 \mathrm{d}x \left(\int_{\partial\mathbb{R}_+^n} |u| \mathrm{d}s\right)^{\frac{2}{n-1}}}{\left(\int_{\partial\mathbb{R}_+^n} u^2 \mathrm{d}s\right)^{\frac{n}{n-1}}} \,\middle|\, u \in C_0^\infty\left(\mathbb{R}_+^n\right), u \not\equiv 0 \right\}.$$

The computation problem of the exact value of $\widetilde{A}_0(n)$ still remains open.

For compact Riemannian manifolds with boundary, Humbert, also, studied in [8] the trace Nash inequality.

On smooth compact n-dimensional, $n \geq 2$, Riemannian manifolds with boundary, for all $u \in C^\infty(M)$, consider the following trace Nash inequality

$$\left(\int_{\partial M} u^2 \mathrm{d}S_g\right)^{\frac{n}{n-1}} \leqslant \left(\tilde{A} \int_M |\nabla u|_g^2 \,\mathrm{d}V_g + \tilde{B} \int_{\partial M} u^2 \mathrm{d}S_g\right) \left(\int_{\partial M} |u| \,\mathrm{d}S_g\right)^{\frac{2}{n-1}}. \tag{5}$$

The best constant for the above inequality is defined as

$$\widetilde{A}_{\mathrm{opt}}(M) = \inf\left\{\tilde{A} > 0 : \exists \tilde{B} > 0 \text{ s.t. (5) is true } \forall u \in C^\infty(M)\right\}.$$

It was proved in [8] that $\widetilde{A}_{\mathrm{opt}}(M) = \widetilde{A}_0(n)$, and (5) with its optimal constant $\widetilde{A} = \widetilde{A}_0(n)$ is always valid.

In this paper we prove that, when the functions are invariant under the action of an isometry group, all orbits of which are of infinite cardinal, the Nash inequalities can be improved, in the sense that we can get a higher critical exponent.

More precisely we establish:

(A) The best constant for the *Nash inequality* on compact Riemannian manifolds with boundary, invariant under the action of an arbitrary compact subgroup G of the isometry group $Is(M, g)$, where all the orbits have infinite cardinal, and
(B) The best constant for the *trace Nash inequality* on compact Riemannian manifolds with boundary, invariant under the action of an arbitrary compact subgroup G of the isometry group $Is(M, g)$, where all the orbits have infinite cardinal.

These best constants are improvements over the classical cases due to the symmetries which arise and reflect the geometry of the manifold.