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The economics of forestry and a set-valued turnpike of the classical type[☆]

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ABSTRACT

In recent work, the authors set classical turnpike theory in the context of the economics of forestry, as developed by Mitra and Wan, and presented two far-reaching results. In this paper, we present a conceptual generalization that takes this theory and configures it around a set in the space of forest configurations rather than around the *golden-rule* forest configuration. Our set-valued analysis hinges on periodicity and yields the earlier results as corollaries under a non-interiority condition on the felicity function that shrinks the set to the point. The question that we pose, and answer, has obvious relevance to more general contexts and, in particular, to turnpike theory as developed by Samuelson, Gale, McKenzie, and their followers.

1. Introduction

In the first issue of the journal *Nonlinear Analysis*, Paul Samuelson [1] presented a periodic turnpike theorem. From the perspective of classical turnpike theory, as recently delineated in [2,3], it is exactly the type of theorem that one would expect and want. Given initial and final capital stocks over a sufficiently large but finite horizon, an optimal intertemporal resource allocation program stays arbitrary close, most of the time, to the solution of an infinite horizon optimal intertemporal resource allocation program, even when the felicity function is subject to periodic oscillations, which is to say, even when the relevant turnpike is periodic. As explained in [4], and subsequently in [5,2], given the differing time horizons, and therefore differing programming problems that are involved, the question is more subtle than that of the asymptotic convergence of the solution of an infinite horizon variational problem, or of the continuity of its solution with respect to initial stocks. Periodicity, as in [1], only adds to the complexity of the question, though the question is now being phrased in the non-stationary register rather than the classical stationary one.

It is of interest that in the same year that he published his periodic turnpike theorem, Samuelson [6] turned his attention to the economics of forestry, and asked whether a profit maximizing firm would be led by market conditions to produce maximal long-run sustainable timber yields. The registers of the two enquiries were different: the first was in the context of a Ramseyian planner [7] concerned with long-term societal interest whereas the second concerned competition and the price theory of the firm. A decade was to pass before Mitra and Wan [8,9] reconfigured Samuelson's forestry problem

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into a Ramseyian planning exercise, and recast it into what has subsequently come to be known as the Gale–McKenzie theory of intertemporal resource allocation [10,11]. In particular, they showed that in a setting with linear felicity functions, periodicity is the rule rather than the exception, and that optimal solutions to the undiscounted infinite horizon problem oscillate around the forest configuration producing maximal sustainable timber yield irrespective of the given initial configuration. It is only in the case of strictly concave undiscounted felicity functions that we obtain convergence to this maximal sustainable timber yield forest configuration. And so, as a first step, it is natural to limit oneself to the strictly concave case, and ask whether there is a viable and robust turnpike theory that can be constructed. This is to ask whether for arbitrary given initial and final forest configurations, optimal forest management would dictate staying arbitrarily near the golden-rule forest configuration for most of a large but finite planning horizon. This question has recently been answered in [12,3].

However, given Samuelson's work, it is surely worthwhile to take another look at the case of linear felicities. Here, there is a continuum of periodic optimal paths, one for each initial forest configuration, and there is no question of a turnpike theory of the classical type. Indeed, there is no single candidate for possible service as a turnpike for large but finite optimal programs starting from an arbitrary initial condition. However, one can proceed by taking the given initial configuration x_0 as a datum of the analysis, focus on the infinite horizon periodic optimal program that starts from it, and ask whether this program is a turnpike for all programs starting from x_0 and ending at $x_T \neq x_0$. In other words, does the T-period program starting from x_0 and ending at x_T for arbitrarily large T get onto the infinite horizon program also starting from x_0, and stay on it until the last few periods to end at x_T? To be sure, this is not a question within the ambit of what is seen as classical turnpike theory in [2]; it is rather a version of what McKenzie [13] terms the *early* turnpike. (He refers to asymptotic convergence of optimal programs as the *late* turnpike.) However, there is an antecedent question that can be asked in the light of the fact that the two theorems presented in [3] do somewhat more than simply provide a classical turnpike theory for the Mitra–Wan tree farm in the strictly concave case; they bypass the restricting bipolar dichotomy of linear and strictly concave functions, and consider instead a setting where the felicity functions are assumed only to be concave.

The point of departure of this paper lies in the unification theorem of [14]. In the context of non-differentiable, upper semi-continuous functions that are supportable at the golden-rule stock, they present a *condition on the primitives of the model* which is necessary and sufficient for all good programs to converge to this stationary stock, and thereby allow a robust turnpike theory to be constructed. This condition can be simply stated: it is the requirement that the solution to a finite dimensional optimization problem be the singleton golden-rule stock. An easier to check *non-interiority condition* proves to be sufficient but not necessary: the golden-rule stock is not an interior point of the convex hull of the set of zeros of the *discrepancy function* corresponding to the (stationary) felicity function, which is to say, the function of the difference between the felicity and its linear affine supporting function at the golden-rule stock. In the particular case of a concave felicity function, the *non-interiority condition* becomes also necessary. This then leads immediately to the situation where the necessary and sufficient condition does *not* hold and to the observation that in such a setting, the solution to the optimization problem mentioned above, say V, is exactly the set of initial configurations with periodic optimal programs—and indeed, depending on the felicity function, can become arbitrarily small. The question then naturally arises as to the characteristics of the optimal programs that do not start from, or end at, configurations in this set V. This question has not been posed in the literature before, and we answer it here.

The conceptual innovation pursued in this paper then needs re-emphasis. It is to construct a set-valued turnpike theory of the classical type. Rather than an investigation of a particular periodic program as a possible candidate for a turnpike, as in [1], it is to bunch the entire continuum of periodic programs together, and investigate the entire set V as a possible candidate for a turnpike. Rather than the metaphor of a freeway or turnpike, as originally described in [4] and quoted in [13], the relevant metaphor here is that of an air-lane or sea-lane that takes on journeys that are long enough, even though it may not be the most direct route; and somewhat more relevantly here, even though within the lane, there are many possible routes and which particular route is taken on one occasion is not the most relevant consideration. From a technical point of view, it is then to substitute a set for a point, and to obtain a non-trivial generalization of the theory that reduces to the standard one when the set V shrinks to a point and the non-interiority condition is automatically activated.

We have already clarified that even though inspired by Samuelson's periodic turnpike theorem, ours is not the same as his. Our set-valued turnpike, in keeping with the stationary assumptions of our model, is not periodic, even though individual optimal paths within it are. It is also worth noting how our *set-valued* turnpike theorem differs from Keeler's *twisted* turnpike theorem, as in [15,16], and from McKenzie's *neighborhood* turnpike theorem, as in [17,18]. Keeler's results can best be seen as antecedent to those of Samuelson's: he works with the original turnpike conception without any consumption and in which labor is a produced input, one rooted in the von Neumann model [19], and allows for non-stationarity in the technology rather than in the felicities, and a non-stationarity of a more general type than periodicity. As emphasized above, the primary interest in the results that we report possibly hinges on the fact that they pertain to a stationary environment. McKenzie's neighborhood turnpike theorem also pertains to a stationary environment, and for the canonical model that goes beyond that of [19] or [4], but it also has a nature totally different to that of ours. His result lies on the interface between the discounted and undiscounted settings, rather than in the undiscounted setting, as for the case that we consider here. For any arbitrarily small neighborhood of the golden-rule stock, his theorem shows the existence of a threshold discount factor such that for all discount factors greater than it, and for any initial capital stock, the program optimal for the chosen discount factor eventually lies in the neighborhood. If one adheres to the formulation of classical turnpike theory as delineated in [2], this is really a type of uniform asymptotic convergence theorem, a late turnpike type of result, with the uniformity, and the