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Existence and decay of solutions in full space to Navier–Stokes equations with delays

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ABSTRACT

We consider the Navier–Stokes equations with delays in $\mathbb{R}^n$, $2 \leq n \leq 4$. We prove existence of weak solutions when the external forces contain some hereditary characteristics and uniqueness when $n = 2$. Moreover, if the external forces satisfy a time decay condition we show that the solution decays at an algebraic rate.

1. Introduction

Navier–Stokes equations have been extensively studied over the last decades due to their importance in fluid mechanics and turbulence. Recently, Caraballo and Real [1] initiated the study of the Navier–Stokes equations with a forcing term which contains some hereditary characteristics. This situation may appear, for instance, when we want to control the system by applying a force which takes into account not only its present state but the history of the solutions.

It is also worth pointing out that partial differential equations with delays arise from various applications, like mathematical biology, climate models, and many others (see e.g. [2,3]). For instance, Navier–Stokes equations with a hereditary viscous term which depends on the past history arise in the dynamics of non-Newtonian fluids and also as viscoelastic models for the dynamics of turbulence statistics in Newtonian fluids (e.g. [4]).

Some further models in fluid mechanics which take into account the history of the solution have also been considered. We can cite Caraballo et al., that considered the 3D Lagrangian averaged Navier–Stokes (LANS-α) model with delay [5] and a 3D α-Navier–Stokes model with delay in [6,7]; Medjo [8,9] and Wan and Duan [10] for the multi-layer quasi-geostrophic ocean model with delay; Medjo [11] for the primitive equations with delay; Liu [12] and Tang and Wan [13] for a time-delayed Burgers equation.

Concerning the Navier–Stokes equations with delay in the external force, the existence and uniqueness of solutions were investigated by Caraballo and Real in [1]. The asymptotic behaviour of the solutions for the two dimensional case was studied by the same authors in [14] and the existence of attractors in [15]. Taniguchi [16] discussed the existence and the exponential behaviour of solutions to Navier–Stokes equations with various types of time-delayed external forces for both 2D and 3D. Planas and Hernández [17] proved similar results, but with delay in the advective term as well as in the external forces. All of the results just mentioned were proven for bounded domains. Some of the 2D results were extended to unbounded domains with boundary satisfying the Poincaré inequality by Garrido and Marín-Rubio [18] and by Marín-Rubio and Real [19]. As far as we know, none of these results have been proven when the equations are considered in the full plane or space.

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The goal of this work is to extend these results by considering the Navier–Stokes equations with delays in $\mathbb{R}^n$, with $2 \le n \le 4$. Namely, we are interested in proving the existence and estimating the decay of solutions to the equations

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \Delta\mathbf{u} = \mathbf{f} - \nabla p + \mathbf{g}_{\tau_h\mathbf{u}}, \quad t > 0,\\
&\mathbf{u}(x,0) = \mathbf{u}_0(x),\\
&\mathbf{u}(x,t) = \phi(x,t), \quad -h < t < 0,\ h > 0,\\
&\nabla\cdot\mathbf{u} = 0, \quad t > 0,
\end{aligned} \tag{1.1}$$

where $\mathbf{u}$ is the velocity and p is the associated pressure. We use the notation $\tau_h\mathbf{u} = \tau_h\mathbf{u}(x,t) = \mathbf{u}(x,t-h)$ and hence our delayed external force is

$$\mathbf{g}_{\tau_h\mathbf{u}} = \mathbf{g}_{\tau_h\mathbf{u}}(x,t) = \mathbf{g}\left(\mathbf{u}(x,t-h),t\right).$$

We first introduce the following function spaces, which are usual in the study of Navier–Stokes equations:

$$\begin{aligned}
&H_0^1(\mathbb{R}^n) = \text{closure of } C_0^\infty(\mathbb{R}^n) \text{ in the norm } \left(\int_{\mathbb{R}^n}|\nabla\mathbf{u}|^2\mathrm{d}x\right)^{1/2},\\
&\mathcal{V} = \{\mathbf{u}\in C_0^\infty(\mathbb{R}^n) : \nabla\cdot\mathbf{u} = 0\},\\
&H = \text{closure of } \mathcal{V} \text{ in } L^2(\mathbb{R}^n),\\
&V = \text{closure of } \mathcal{V} \text{ in } H_0^1(\mathbb{R}^n).
\end{aligned}$$

As usual, X' will denote the dual space dual of X. To denote the Fourier Transform of a function φ we will use either $\mathcal{F}(\varphi)$ or $\widehat{\varphi}$. Throughout we will use C to denote an arbitrary constant which may change line to line.

We now establish the conditions that the external forces $\mathbf{f}$ and $\mathbf{g}$ must satisfy in order to prove our results. These are

$$\begin{aligned}
&\mathbf{f}\in L^1\cap L^\infty(\mathbb{R}_+; W^{-1,1}),\\
&\nabla\cdot\mathbf{f} = 0,\\
&\mathbf{f}\in L^2(0,T;V'), \quad T > 0,\\
&\|\mathbf{f}(t)\|_{L^2} \le K(1+t)^{-\left(\frac{n}{2}+1\right)}, \quad t > 0,
\end{aligned} \tag{1.2}$$

and

$$\begin{aligned}
&\mathbf{g}(x,t) \text{ is locally Lipschitz continuous in } x,\\
&\mathbf{g} = \nabla G, \qquad |G(x,t)| \le K_1|x|^2, \quad x\in\mathbb{R}^n,\ t > 0,\\
&|\mathbf{g}(x,t)| \le K_2(1+t)^{-\left(\frac{n}{2}+1\right)}|x|, \quad x\in\mathbb{R}^n,\ t > 0.
\end{aligned} \tag{1.3}$$

Remark 1.1. Conditions on $\mathbf{f}$ are very similar to the usual ones asked for when proving uniform decay in the case of the Navier–Stokes equations, see [20–22]. The time decay condition on $\mathbf{g}$ is similar to the one for $\mathbf{f}$, while the Lipschitz-like conditions on $\mathbf{g}$ are similar to those in [1,14,16]. An example of a function G for which conditions (1.3) hold is $G(x,t) = C(1+t)^{-\left(\frac{n}{2}+1\right)}\log(1+|x|^2)$. □

We now state the main result of this article.

Theorem 1.1. *Let $\mathbf{u}_0\in L^1\cap H$ and $\phi\in L^\infty(-h,0;H)$ and let $\mathbf{f}$ and $\mathbf{g}$ satisfy (1.2) and (1.3) respectively. Then for any $T > 0$ there exists a weak solution $\mathbf{u}$ to (1.1) with*

$$\mathbf{u}\in L^\infty(-h,T;H)\cap L^2(0,T;V),$$

and

$$\|\mathbf{u}(t)\|_{L^2} \le C\,(1+t)^{-\frac{n}{4}}, \quad t > 0$$

where $C = C_n(\|\mathbf{u}_0\|_{L^1}, \|\mathbf{u}_0\|_{L^2}, \|\phi\|_{L^\infty_t L^2_x})$. When $n = 2$, this solution is unique.

Remark 1.2. The regularity and decay rates are the same as for the usual Navier–Stokes equations, see the results in [23,22,24]. □

Remark 1.3. Theorem 1.1 also holds for variable delays. For the sake of clarity, we state this result in Section 4, where we indicate the changes in the proof of Theorem 1.1 needed to establish this result. □