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A hybrid scheme for finite families of equilibrium, variational inequality and fixed point problems

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ABSTRACT

We introduce an iterative process for finding an element in the common fixed point set of finite family of closed relatively quasi-nonexpansive mappings, common solutions of finite family of equilibrium problems and common solutions of finite family of variational inequality problems for monotone mappings in Banach spaces. Our theorem extends and unifies most of the results that have been proved for this important class of nonlinear operators.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Banach space E with dual E^*. A mapping $A : D(A) \subset E \to E^*$ is said to be *monotone* if for each $x, y \in D(A)$, the following inequality holds:

$$\langle x - y, Ax - Ay\rangle \geq 0. \tag{1.1}$$

A is said to be *γ-inverse strongly monotone* if there exists a positive real number γ such that

$$\langle x - y, Ax - Ay\rangle \geq \gamma \|Ax - Ay\|^2, \quad \text{for all } x, y \in D(A). \tag{1.2}$$

If A is γ-inverse strongly monotone, then it is *Lipschitz continuous* with constant $\frac{1}{\gamma}$, that is, $\|Ax - Ay\| \leq \frac{1}{\gamma}\|x - y\|$, for all $x, y \in D(A)$, and hence uniformly continuous. Clearly, the class of monotone mappings include the class of γ-inverse strongly monotone mappings.

Suppose that A is a monotone mapping from C into E. The variational inequality problem is formulated as follows

$$\text{find a point } u \in C \text{ such that } \langle v - u, Au\rangle \geq 0, \quad \text{for all } v \in C. \tag{1.3}$$

The solution set of the variational inequality problem is denoted by $VI(C, A)$.

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Variational inequalities were initially studied by Stampacchia [1,2] and ever since have been widely studied in general Banach spaces (see, e.g., [3–6]). Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding point $u \in C$ such that $0 \in Au$.

Let E be a real Banach space with dual E^*. We denote by J the *normalized duality mapping* from E into 2^{E^*} which is defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle ., . \rangle$ denotes the duality pairing. It is well known that if E is smooth then J is single-valued and demicontinuous, and if E is uniformly smooth then J is uniformly continuous on bounded subsets of E. Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single-valued, one-to-one, surjective and it is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see, e.g., [7,8]). We note that in a Hilbert space H, the mapping J is the identity operator.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for any } x, y \in E, \tag{1.4}$$

was studied by Alber [9], Kamimula and Takahashi [4] and Riech [10]. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2 \quad \text{for any } x, y \in E. \tag{1.5}$$

We remark that $\phi(x, y) = 0$ if and only if $x = y$ (see, e.g., [11]). Moreover, observe that in a Hilbert space H, (1.4) reduces to $\phi(x, y) = \|x - y\|^2$, for any $x, y \in H$.

Let E be a reflexive, strictly convex and smooth Banach space and let C be a nonempty, closed and convex subset of E. The *generalized projection mapping*, which was introduced by Alber [9], is the mapping $\Pi_C : E \to C$, that assigns an arbitrary point $x \in E$ to the minimizer, $\bar{x}$, of $\phi(., x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y, x), y \in C\}. \tag{1.6}$$

We have the following result.

Lemma 1.1 ([9]). *Let C be a nonempty, closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \min\{\phi(z, x) : z \in C\}$.*

Let C be a nonempty, closed convex subset of E and let T be a mapping from K into itself. We denote by $F(T)$ the fixed point set of T. A point p in C is said to be an *asymptotic fixed point of* T (see [10]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for each $x, y \in C$ and is called *relatively nonexpansive* [12,6] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mapping was studied in [13]. A mapping T is said to be *ϕ-expansive* if $\phi(Tx, Ty) \le \phi(x, y)$ for all $x, y \in C$. A mapping T is said to be *relatively quasi-nonexpansive* if $F(T) \ne \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. A mapping T is said to be *closed* if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Tx_n = y$ imply that $Tx = y$.

Remark 1.2. The class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings (see [13]) which requires the strong restriction: $F(T) = \hat{F}(T)$.

If E is smooth, strictly convex and reflexive Banach space, and $A \subset E \times E^*$ is a continuous monotone mapping with $A^{-1}(0) \ne \emptyset$ then it is proved in [14] that the resolvent $J_r := (J + rA)^{-1}J$, for $r > 0$ is relatively quasi-nonexpansive. Moreover, if $T : E \to E$ is relatively quasi-nonexpansive then using the definition of ϕ one can show that $F(T)$ is closed and convex (see [11]).

In 1953, Mann [15] introduced the iteration sequence $\{x_n\}_{n\in\mathbb{N}}$ which is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1.7}$$

where the initial element $x_0 \in C$ is arbitrary and $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers in $[0, 1]$, for nonexpansive mapping T. One of the fundamental convergence results is proved by Reich [16]. He proved that the sequence (1.7) converges weakly to a fixed point of T, under appropriate conditions on $\{\alpha_n\}_{n\in\mathbb{N}}$ and the Banach space E. In an infinite dimensional Hilbert space, the Mann iteration can conclude only weak convergence (see [17,18]). Attempts to modify the Mann iteration method (1.7) so that strong convergence is guaranteed have recently been made. Bauschke and Combettes [19] (see also [20]) proposed the following modification of the Mann iteration method (1.7):

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n \ge 0, \end{cases} \tag{1.8}$$