Nonlinear Analysis 74 (2011) 286–297

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

# Global existence of solutions to a parabolic–parabolic system for chemotaxis with weak degradation

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ARTICLE INFO

*Article history:*
Received 28 April 2010
Accepted 23 August 2010

*MSC:*
35K51
35K57
35K92
92C17

*Keywords:*
Chemotaxis
Logistic source
Parabolic–parabolic system
Global existence

ABSTRACT

We study the global existence of solutions to a parabolic–parabolic system for chemotaxis with a logistic source in a two-dimensional domain, where the degradation order of the logistic source is weaker than quadratic. We introduce nonlinear production of a chemoattractant, and show the global existence of solutions under certain relations between the degradation and production orders.



## 1. Introduction

Chemotaxis is the movement of biological individuals towards (or away from) a chemoattractant (or chemorepellent). It plays a crucial role in the study of chemotactic phenomena such as wound healing, cancer growth and leukocyte movement [1,2]. Bacterial species such as *E. coli* exhibit chemotaxis with respect to the chemoattractant aspartate which they themselves produce. Budrene and Berg [3,4] found that *E. coli* formed remarkable macroscopic regular patterns in their colony resulting from the interplay between chemotaxis, proliferation and the reduction in numbers due to death (we refer to the proliferation and reduction in numbers together simply as growth). In fact, growth over time is necessary for the colony patterns to be developed; sufficient time is required for several generations of *E. coli* to proliferate and die, thereby generating the pattern formation processes.

Many mathematical models have been proposed to elucidate the mechanisms behind these patterns generated by bacterial species. Several of them are based on partial differential equations including terms for chemotaxis and growth. In the equations, chemotaxis can be expressed as a *negative diffusion* that is a directed flux toward the gradient of chemical concentration, and growth as a saturating logistic function. Mimura and Tsujikawa [5] proposed a chemotaxis-growth system of equations, and a simplified version is of the form:

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doi:10.1016/j.na.2010.08.044

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u) & \text{in } \Omega \times (0, \infty), \\ \tau \dfrac{\partial v}{\partial t} = \Delta v - v + g(u) & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{E}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, and $\partial/\partial n$ denotes the derivative with respect to the outer normal of $\partial\Omega$. The function $u(x, t)$ is the population density of biological individuals at the position $x \in \Omega$ and time $t \in [0, \infty)$, and $v(x, t)$ is the concentration of chemical substance that is produced by the individuals. The functions $f(u)$ and $g(u)$ are the growth of $u$ and the production of $v$, respectively; the coefficients $\chi$ and $\tau$ are positive constants.

Several types of functions have been proposed for $f(u)$ and $g(u)$ [6,7]. In [5] it was assumed that $f(u)$ was a cubic logistic function and $g(u)$ was a linear production ($g(u) = g_0 u$). When no growth ($f(u) = 0$) and $g(u) = g_0 u$ are assumed, the system (E) is equivalent to a classical Keller–Segel system [8] in which a finite-time blow-up with a $\delta$-function singularity of $u$ occurs when $\|u_0\|_{L^1}$ (or $\chi$) is sufficiently large [9,10]. On the other hand, if $f(u) = u(1 - u)$ and $g(u) = g_0 u$, then blow-up does not occur and global existence of solutions is assured for any nonnegative initial function $(u_0, v_0) \in L^2(\Omega) \times H^{1+\varepsilon_0}(\Omega)$, $0 < \varepsilon_0 < 1/2$, even if $\|u_0\|_{L^1}$ and $\chi$ are large [11]. From these results, we find that when the degradation order of $f(u)$ is two (and $g(u)$ is linear), the blow-up of solutions is prevented.

Nagai [12] studied the Keller–Segel system ($f(u) = 0$ and $g(u) = g_0 u$) in the situation where the chemoattractant diffuses very quickly, and proved the existence of a threshold number $8\pi/(g_0\chi)$ of $\|u_0\|_{L^1}$ between global existence and finite-time blow-up of solutions. For further study we refer the reader to [13–16] (see also [17,18]). For the system (E) the situation corresponds to the case when $\tau$ tends to 0. When $\tau = 0$ the second equation reduces to an elliptic equation; we then refer to the system (E) with $\tau = 0$ and $g(u) = g_0 u$ as a parabolic–elliptic system ($E_0$). Winkler [19] studied the system ($E_0$) in an $n$-dimensional smooth bounded domain under the assumption that $-\nu(u + u^\alpha) \le f(u) \le \mu_0 - \mu_1 u^\alpha$ for some positive constants $\nu$, $\mu_0$ and $\mu_1$. This assumption implies that $f(u)$ has a degradation order of $\alpha$. Winkler introduced the notion of a *very weak solution* and showed global existence of solutions for small $u_0 \in L^1(\Omega)$ with $\alpha > \max\{n/2, 2 - 1/n\}$ and $\mu_1/\mu_0$ large (cf. [20] and Appendix B). Then, roughly speaking, for the two-dimensional system ($E_0$) with $\alpha > 3/2$ and large $\mu_1/\mu_0$, global existence of solutions is assured for small initial functions.

We henceforth assume that the functions $f(u)$ and $g(u)$ are smooth functions satisfying the following estimates for the degradation and production orders:

$$\begin{cases} f(u) \le 1 - \mu u^\alpha \quad \text{for } u \ge 0, \\ 0 \le g(u) \le (u + 1)^\beta \quad \text{and} \quad 0 \le g'(u) \le \beta(u + 1)^{\beta - 1} \quad \text{for } u \ge 0, \end{cases} \tag{1}$$

where $\mu > 0$, $\alpha \ge 1$ and $\beta > 0$. In the present paper, we will study global existence of solutions for the parabolic–parabolic system (E) with (1). Indeed, under certain relations between $\alpha$ and $\beta$, we will show that for any nonnegative initial function $(u_0, v_0) \in L^2(\Omega) \times H^2_N(\Omega)$ and any positive coefficient $\chi$, there exists a unique nonnegative global solution $(u, v)$ in the function space:

$$\begin{cases} 0 \le u \in \mathcal{C}^1((0, \infty); L^2(\Omega)) \cap \mathcal{C}([0, \infty); L^2(\Omega)) \cap \mathcal{C}((0, \infty); H^2_N(\Omega)), \\ 0 \le v \in \mathcal{C}^1((0, \infty); H^2_N(\Omega)) \cap \mathcal{C}([0, \infty); H^2_N(\Omega)) \cap \mathcal{C}((0, \infty); H^4_{N^2}(\Omega)). \end{cases} \tag{2}$$

Moreover, we will show the boundedness of solutions, namely that

$$\|u(t)\|_{L^2} + \|v(t)\|_{H^2} \le \psi(\|u_0\|_{L^2} + \|v_0\|_{H^2}), \quad t \ge 0 \tag{3}$$

with some increasing function $\psi(\cdot)$. Here, the spaces $H^2_N(\Omega)$ and $H^4_{N^2}(\Omega)$ are Sobolev spaces with the boundary conditions, defined precisely in Section 2.

By analogy with the results for ($E_0$), we are then concerned with the global existence of solutions to the system (E) for the case of $\alpha < 2$ (and also $\alpha \ge 2$) with $\beta = 1$. The method of showing global existence is to estimate dissipative energy of the system in the space $L^2(\Omega) \times H^2_N(\Omega)$, e.g. as used in [11]. To apply the method, however, may require the relations $\beta \le \alpha/2$ and $\beta \le \alpha - 1$ (Theorems 1–3). In our arguments, therefore, a sub-linear production $\beta < 1$ is needed for $\alpha < 2$; and conversely we find that $\alpha > 2$ is needed for estimates with a super-linear production $\beta > 1$. These restrictions on $\alpha$ and $\beta$ do not cover the case when $3/2 < \alpha < 2$ and $\beta = 1$ of [19], although our global existence results can be applied to other chemotaxis systems including growth and production satisfying the order estimates (1). As possible nonlinear forms of $g(u)$, we consider the saturating function $g(u) = g_0 u/(1 + g_1 u)$, as used in the nonlinear signal kinetics model [6, Section 1.2] and the model for snake integument patterns [2, Section 4.12], and also a quadratic production function $g(u) = g_0 u^2$, as used for another type of chemotaxis-growth system [2, Section 5.2]. See the review article by Hillen and Painter [6], which includes this topic (see also the review article by Tindall et al. [7]).

The main theorem of this paper can be stated as follows: