Nonlinear Analysis 74 (2011) 298–316

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

The existence and dynamic properties of a parabolic nonlocal MEMS equation

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ARTICLE INFO

Article history:
Received 5 May 2010
Accepted 23 August 2010

MSC:
primary 35B40
secondary 35B05
35K50
35K20

Keywords:
Nonlocal MEMS
Pull-in voltage
Parabolic nonlocal MEMS
Asymptotic behaviour
Quenching behaviour

ABSTRACT

Let $\Omega \subset \mathbb{R}^n$ be a C^2 bounded domain and $\chi > 0$ be a constant. We will prove the existence of constants $\lambda_N \geq \lambda_N^* \geq \lambda^*(1 + \chi \int_\Omega \frac{dx}{1-w_*})^2$ for the nonlocal MEMS equation $-\Delta v = \lambda/(1-v)^2(1+\chi\int_\Omega 1/(1-v)dx)^2$ in Ω, $v = 0$ on $\partial\Omega$, such that a solution exists for any $0 \leq \lambda < \lambda_N^*$ and no solution exists for any $\lambda > \lambda_N$ where λ^* is the pull-in voltage and w_* is the limit of the minimal solution of $-\Delta v = \lambda/(1-v)^2$ in Ω with $v = 0$ on $\partial\Omega$ as $\lambda \nearrow \lambda^*$. Moreover $\lambda_N < \infty$ if Ω is a strictly convex smooth bounded domain. We will prove the local existence and uniqueness of the solution of the parabolic nonlocal MEMS equation $u_t = \Delta u + \lambda/(1-u)^2(1+\chi\int_\Omega 1/(1-u)dx)^2$ in $\Omega \times (0,\infty)$, $u = 0$ on $\partial\Omega \times (0,\infty)$, $u(x,0) = u_0$ in Ω. We prove the existence of a unique global solution and the asymptotic behaviour of the global solution of the parabolic nonlocal MEMS equation under various boundedness conditions on λ. We also obtain the quenching behaviour of the solution of the parabolic nonlocal MEMS equation when λ is large.

1. Introduction

Microelectromechanical systems (MEMS) are widely used nowadays in many electronic devices including accelerometers for airbag deployment in cars, inkjet printer heads, and devices for the protection of hard disks, etc. The challenge is to build and understand mathematical models and the mechanisms for the various MEMS devices. Recently there has been a lot of study of the equations arising from MEMS by Esposito et al. [1–4], Kavallaris et al. [5], Lin and Yang [6], Ma and Wei [7], Flores et al. [8–10] etc. Interested readers can read the book "Modeling MEMS and NEMS" [11], by Pelesko and Bernstein for the mathematical modeling and various applications of MEMS devices.

In [11] Pelesko and Bernstein model the deflection between the two parallel plates of an electrostatic MEMS device with the equation

$$\begin{cases} -\Delta w = \dfrac{\lambda}{(1-w)^2} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{S_λ}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded C^2 domain. Interested readers can read the papers [2,5,6] for various results on the above equation. In [6] Lin and Yang by using a variational argument derived the following nonlocal MEMS equation:

$$\begin{cases} -\Delta v = \dfrac{\lambda}{(1-v)^2\left(1+\chi\int_\Omega \frac{dx}{1-v}\right)^2} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{S_λ^N}$$

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doi:10.1016/j.na.2010.08.045

for an electrostatic MEMS device with circuit series capacitance that models the deflection between a membrane and an upper plate which is parallel to the plane containing the boundary of the membrane. An interesting property of (S_λ) [2,6] is that there exists a $\lambda^* > 0$, called the pull-in voltage in the literature on MEMS research, such that (S_λ) has a solution for any $0 \le \lambda < \lambda^*$ and no solution exists for any $\lambda > \lambda^*$. Physically this corresponds to the existence of a pull-in voltage such that the membrane and the upper plate in the MEMS device collapse together [6,11], when λ which is proportional to the square of the electric voltage between the membrane and the upper plate is greater than the pull-in voltage λ^*.

In this paper we will study the equation (S_λ^N) and show that (S_λ^N) has similar properties. Let $\chi > 0$. We will study the existence and non-existence of solutions of the corresponding nonlocal parabolic equation (cf. [11,12]),

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \dfrac{\lambda}{(1-u)^2\left(1+\chi\int_\Omega \frac{dy}{1-u(y,t)}\right)^2} & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0 & \text{in } \Omega \end{cases} \qquad (P_\lambda)$$

where $\lambda \ge 0$ is a constant. The above equation also appears in the unpublished preprint "Pull-in voltage and steady states of nonlocal electrostatic MEMS" of Ghoussoub and Guo. We will prove the local existence and uniqueness of the solution of (P_λ). Under some boundedness conditions for λ we prove the existence of a unique global solution and the asymptotic behaviour of the global solution of (P_λ). We prove the quenching behaviour of the solution of (P_λ) for when $u_0 \equiv 0$ on Ω and λ is large. Physically this corresponds to the case where there is no deflection of the plates at the initial time $t = 0$ and the applied voltage is large. We also prove the quenching behaviour of the solution of (P_λ) for when Ω is a ball, u_0 is radially symmetric, and λ is large.

The plan of the paper is as follows. In Section 2 we will prove the existence of constants $\lambda_N \ge \lambda_N^* \ge \lambda^*\left(1+\chi\int_\Omega \frac{dx}{1-w_*}\right)^2$ such that (S_λ^N) has a solution for any $0 \le \lambda < \lambda_N^*$ and (S_λ^N) has no solution for any $\lambda > \lambda_N$. We also prove the boundedness of λ_N for when Ω is a strictly convex smooth bounded domain of $\mathbb{R}^n$. In Section 3 we will prove the local existence and uniqueness of the solution of (P_λ). We also obtain energy estimates for the solution of (P_λ). In Section 4 we prove the global existence and asymptotic behaviour of the global solution of (P_λ) under various boundedness conditions on λ. In Section 5 we prove the quenching behaviour of the solution of (P_λ) for when λ is large.

We will assume that $\Omega \subset \mathbb{R}^n$ is a bounded C^2 domain for the rest of the paper. We start with some definitions. For any $\delta > 0, R > 0$, let $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}$ and $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. We say that w is a solution of (S_λ) (respectively, (S_λ^N)) if $w \in C^2(\Omega) \cap C(\overline{\Omega})$, $0 \le w < 1$ in Ω, satisfies (S_λ) (respectively, (S_λ^N)) in the classical sense.

For any constants $\chi \ge 0, \lambda > 0, f \in C(\overline{\Omega} \times (0,T))$ and

$$u_0 \in L^1(\Omega) \quad \text{with } u_0 \le a \text{ a.e. in } \Omega \tag{1.1}$$

for some constant $0 < a < 1$ we say that u is a solution (respectively, subsolution, supersolution) of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \dfrac{\lambda f}{(1-u)^2\left(1+\chi\int_\Omega \frac{dy}{1-u(y,t)}\right)^2} & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0 & \text{in } \Omega \end{cases}$$

in $\Omega \times (0,T)$ if $u \in C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times (0,T))$, $0 \le u < 1$, satisfies

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\lambda f}{(1-u)^2\left(1+\chi\int_\Omega \frac{dy}{1-u(y,t)}\right)^2} \quad \text{in } \Omega \times (0,T)$$

(respectively, $\le, \ge$) in the classical sense with $u(x,t) = 0$ (respectively, $\le, \ge$) on $\partial\Omega \times (0,T)$,

$$\sup_{\overline{\Omega} \times [0,T']} u(x,t) < 1 \quad \forall 0 < T' < T,$$

and

$$\|u(\cdot,t) - u_0\|_{L^1(\Omega)} \to 0 \quad \text{as } t \to 0. \tag{1.2}$$

Let μ_1 be the first positive eigenvalue and ϕ_1 be the first positive eigenfunction of $-\Delta$ which satisfies $\int_\Omega \phi_1 dx = 1$. For any solution u of (P_λ) we define the quenching time $T_\lambda > 0$ as the time which satisfies

$$\begin{cases} \sup_\Omega u(x,t) < 1 \quad \forall 0 < t < T_\lambda \\ \lim_{t \nearrow T_\lambda} \sup_\Omega u(x,t) = 1. \end{cases}$$

We say that u has a finite quenching time if $T_\lambda < \infty$ and we say that u quenches at time infinity if $T_\lambda = \infty$.