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Locally defined operators in Hölder's spaces

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ABSTRACT

We prove that every local operator acting between two Hölder spaces H_ϕ and H_ψ is a Nemytskij operator and if $H_\phi \not\subseteq H_\psi$ then it is a constant map. Moreover, we show that if K is local and uniformly continuous then it is an affine mapping.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and $\mathcal{G} = \mathcal{G}(I, \mathbb{R})$, $\mathcal{H} = \mathcal{H}(I, \mathbb{R})$ be two classes of functions $\varphi : I \to \mathbb{R}$. A mapping $K : \mathcal{G} \to \mathcal{H}$ is said to be a *locally defined operator*, briefly, a *local operator*, if for any open interval $J \subset \mathbb{R}$ and for any functions $\varphi, \psi \in \mathcal{G}$,

$$\varphi|_{J\cap I} = \psi|_{J\cap I} \Rightarrow K(\varphi)|_{J\cap I} = K(\psi)|_{J\cap I},$$

where $\varphi|_{J\cap I}$ denotes the restriction of φ to $J \cap I$.

The form of locally defined operators strongly depends on the properties of both function spaces $\mathcal{G}$ and $\mathcal{H}$. For instance if $\mathcal{G} = C^m(I)$ and $\mathcal{H} = C^i(I)$, $i = 0, 1, \dots, m$, where m is a nonnegative integer and $C^m(I)$ denotes the space of all m-times continuously differentiable functions (cf. [1–4]), then for all $\varphi \in C^m(I)$,

$$K(\varphi)(x) = h(x, \varphi'(x), \dots, \varphi^{(m-i)}(x)), \quad x \in I,$$

for some function $h : I \times \mathbb{R}^{m-i+1} \to \mathbb{R}$ (for $k = 2, \dots, m$ an operator K is additionally polynomially $(m-k)$-bounded [3]). In particular, if $K : C^0(I) \to C^0(I)$ then K is a Nemytskij (superposition) operator and the generating function (by the Krasnosielskij theorem [5, Theorem 6.3]) is continuous on $I \times \mathbb{R}$.

Similarly, if K maps $C^1(I)$ into $C^1(I)$ then it is also the Nemytskij operator, but, surprisingly enough, in this case its generating function need not be even continuous. In this paper we examine local operators acting between two Hölder spaces. The main result of Section 2 states that any local operator K mapping a Hölder space H_ϕ into H_ψ is a Nemytskij (superposition) operator and, under the assumption that I is compact and $H_\phi \subseteq H_\psi$, that the boundedness assumption of an operator K is necessary for the continuity of its generator (Theorem 2). In the case, where $H_\phi \not\subseteq H_\psi$, it is a constant map. Moreover we show that if a local operator is uniformly continuous, then it is an affine function (Theorem 4).

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2. Local operator

In the sequel $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ denote, respectively, the set of natural numbers, real numbers and the set of positive real numbers. Let a function $\phi : [0, \infty) \to [0, \infty)$ fulfil the following conditions:

1° ϕ is right continuous at 0 and $\phi(0) = 0$;
2° ϕ is strictly increasing;
3° $(0, \infty) \ni t \mapsto \frac{\phi(t)}{t}$ is decreasing.

Remark 1. Note that 1°–3° implies that ϕ is subadditive and continuous.

Indeed, by 3° we get the subadditivity of ϕ [5, Lemma 7.1].
Moreover, the statement 3° implies that for every $t > 0$ there exist $\phi(t-)$, $\phi(t+)$ and

$$\phi(t+) \le \phi(t) \le \phi(t-). \tag{1}$$

On the other hand, by 2°,

$$\phi(t-) \le \phi(t) \le \phi(t+)$$

for all $t \in (0, \infty)$, which together with (1) gives the continuity of ϕ on $(0, \infty)$. Hence, by 1°, ϕ is continuous.

For a function ϕ having properties 1°–3° and an interval $I \subset \mathbb{R}$ by $H_\phi(I) = H_\phi(I, \mathbb{R})$ we denote the Banach space of Hölder functions $\varphi : I \to \mathbb{R}$ equipped with the norm

$$\|\varphi\|_\phi := |\varphi(z)| + \sup_{x,y\in I, x\neq y} \frac{|\varphi(x) - \varphi(y)|}{\phi(|x-y|)},$$

where $z \in I$ is arbitrarily fixed.

Reformulating this definition, we see that $\varphi \in H_\phi(I)$ if and only if there exists a constant $L \in \mathbb{R}_+$ such that

$$|\varphi(x) - \varphi(y)| \le L\phi(|x-y|), \quad x, y \in I. \tag{2}$$

In what follows, if (2) holds true we say that φ is ϕ-Hölder continuous.

Let us notice that if $\phi(t) = t^\alpha$ for some $\alpha \in (0, 1]$ then $H_\alpha(I) = H_\phi(I)$ is the classical Hölder function space and $H_1(I)$ becomes the Banach space of Lipschitz functions.

Theorem 1. *Let $I \subset \mathbb{R}$ be an interval. If a local operator K maps $H_\phi(I)$ into $C^0(I)$, then there exists a unique function $h : I \times \mathbb{R} \to \mathbb{R}$ such that, for all $\varphi \in H_\phi(I)$,*

$$K(\varphi)(x) = h(x, \varphi(x)), \quad x \in I. \tag{3}$$

Proof. We begin by showing that for every $\varphi, \psi \in H_\phi(I)$ and for every $x_0 \in I$ the condition

$$\varphi(x_0) = \psi(x_0) \tag{4}$$

implies that

$$K(\varphi)(x_0) = K(\psi)(x_0). \tag{5}$$

To this end take an arbitrary pair of functions $\varphi, \psi \in H_\phi(I)$ which fulfil (4) and assume first that $x_0 \in \operatorname{int} I$. Since φ, ψ are ϕ-Hölder continuous, there is a constant $L \in \mathbb{R}_+$ such that

$$|\varphi(x) - \varphi(y)| \le L\phi(|x-y|), \qquad |\psi(x) - \psi(y)| \le L\phi(|x-y|); \quad x, y \in I. \tag{6}$$

Define a function $\gamma : I \to \mathbb{R}$ by

$$\gamma(x) = \begin{cases} \varphi(x), & x \le x_0,\ x \in I \\ \psi(x), & x > x_0,\ x \in I. \end{cases}$$

To show that $\gamma \in H_\phi(I)$, take $x, y \in I$ and assume that $x < x_0$ and $y > x_0$. Then

$$|x-y| = |x-x_0| + |y-x_0|. \tag{7}$$

In view of (4), (6) and by the triangle inequality,

$$\begin{aligned} |\gamma(x) - \gamma(y)| &\le |\gamma(x) - \gamma(x_0)| + |\gamma(y) - \gamma(x_0)| \\ &= |\varphi(x) - \varphi(x_0)| + |\psi(y) - \psi(x_0)| \\ &\le L\phi(|x-x_0|) + L\phi(|y-x_0|). \end{aligned}$$