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Coupled fixed point theorems for a generalized Meir–Keeler contraction in partially ordered metric spaces

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ABSTRACT

Let X be a non-empty set and $F : X\times X \to X$ be a given mapping. An element $(x, y) \in X\times X$ is said to be a coupled fixed point of the mapping F if $F(x, y) = x$ and $F(y, x) = y$. In this paper, we consider the case when X is a complete metric space endowed with a partial order. We define generalized Meir–Keeler type functions and we prove some coupled fixed point theorems under a generalized Meir–Keeler contractive condition. Some applications of our obtained results are given. The presented theorems extend and complement the recent fixed point theorems due to Bhaskar and Lakshmikantham [T. Gnana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 1379–1393].

1. Introduction

The Banach contraction principle [1] is a classical and powerful tool in nonlinear analysis and has many generalizations; see [2–9] and others. Recently, Ran and Reurings [10], Bhaskar and Lakshmikantham [11], Nieto and Lopez [12], Agarwal, El-Gebeily and O'Regan [13] and Lakshmikantham and Ćirić [14] presented some new results for contractions in partially ordered metric spaces (see also [15–22]).

In [11], Bhaskar and Lakshmikantham introduced the notion of a coupled fixed point. More precisely, let X be a non-empty set and $F : X \times X \to X$ be a given mapping. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping F if

$$F(x, y) = x, \qquad F(y, x) = y.$$

They also introduced the notion of mixed monotone mapping. If $(X, \leq)$ is a partially ordered set, the mapping F is said to have the mixed monotone property if

$$x_1, x_2 \in X, \qquad x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y), \quad \forall y \in X$$

and

$$y_1, y_2 \in X, \qquad y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2), \quad \forall x \in X.$$

The main theoretical results of Bhaskar and Lakshmikantham in [11] are the following two coupled fixed point theorems.

Theorem 1.1 (*Bhaskar and Lakshmikantham [11]*). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)] \quad \text{for each } x \geq u \text{ and } y \leq v. \tag{1}$$

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If there exist $x_0, y_0 \in X$ such that

$$x_0 \le F(x_0, y_0) \quad \text{and} \quad y_0 \ge F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$F(x, y) = x \quad \text{and} \quad F(y, x) = y.$$

Theorem 1.2 (*Bhaskar and Lakshmikantham [11]*). *Let $(X, \le)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property:*

(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all n,*
(ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \le y_n$ for all n.*

Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \le \frac{k}{2}[d(x, u) + d(y, v)] \quad \text{for each } x \ge u \text{ and } y \le v.$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \le F(x_0, y_0) \quad \text{and} \quad y_0 \ge F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$F(x, y) = x \quad \text{and} \quad F(y, x) = y.$$

Bhaskar and Lakshmikantham [11] have also discussed the problems of a uniqueness of a coupled fixed point and applied their theorems to problems of the existence and uniqueness of solution for a periodic boundary value problem.

In [5], Meir and Keeler generalized the well known Banach fixed point theorem [1]as follows.

Theorem 1.3 (*Meir–Keeler [5]*). *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that*

$$x, y \in X, \quad \varepsilon \le d(x, y) < \varepsilon + \delta(\varepsilon) \Rightarrow d(Tx, Ty) < \varepsilon. \tag{2}$$

Then, T admits a unique fixed point $\xi \in X$ and for all $x \in X$, the sequence $\{T^n x\}$ converges to ξ.

In this paper, we extend and complement Theorems 1.1 and 1.2 of Bhaskar and Lakshmikantham [11] by replacing the contraction hypothesis (1) by a generalized Meir–Keeler contraction. The uniqueness of a coupled fixed point is also discussed. As applications, inspired with works of Branciari [3] and Suzuki [7], we obtain some coupled fixed point theorems for mappings satisfying a general contractive condition of integral type in complete partially ordered metric spaces.

2. Main results

2.1. Preliminaries and definitions

We start by introducing some definitions.

Definition 2.1. Let $(X, \le)$ be a partially ordered set and $F : X \times X \to X$ be a given mapping. We say F has the mixed strict monotone property if

$$x_1, x_2 \in X, \qquad x_1 < x_2 \Rightarrow F(x_1, y) < F(x_2, y), \quad \forall y \in X$$

and

$$y_1, y_2 \in X, \qquad y_1 < y_2 \Rightarrow F(x, y_1) > F(x, y_2), \quad \forall x \in X.$$

Definition 2.2. Let (X, d) be a partially ordered metric space and $F : X \times X \to X$ be a given mapping. We say F is a generalized Meir–Keeler type function if for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$x \ge u,\ y \le v, \quad \varepsilon \le \frac{1}{2}[d(x, u) + d(y, v)] < \varepsilon + \delta(\varepsilon) \Rightarrow d(F(x, y), F(u, v)) < \varepsilon. \tag{3}$$

The following Proposition shows us that (1) is a particular case of (3).

Proposition 2.1. *Let (X, d) be a partially ordered metric space and $F : X \times X \to X$ be a given mapping. If (1) is satisfied, then F is a generalized Meir–Keeler type function.*

Proof. Suppose that (1) is satisfied. For all $\varepsilon > 0$, we check easily that (3) is satisfied with $\delta(\varepsilon) = \left(\frac{1}{k} - 1\right)\varepsilon$. $\square$