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Overview of differential equations with non-standard growth

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ABSTRACT

Differential equations with non-standard growth have been a very active field of investigation in recent years. In this survey we present an overview of the field, as well as several of the most important results. We consider both existence and regularity questions. Finally, we provide a comprehensive list of papers published to date.

I. Introduction

A detailed introduction to this article is given in this section.

1. Overview of the history and the article

Differential equations with non-standard growth and corresponding function spaces with variable exponents have been a very active field of investigation in recent years. We counted over 300 publications from more than 100 authors this decade. The theory of variable exponent Lebesgue and Sobolev spaces has been surveyed in [1,2]; see also the upcoming monograph [3]. Mingione [4] has written an extensive exposition of regularity theory, including the non-standard growth case; Fan [5] summarized some results of his research group on the existence and multiplicity of solutions of eigenvalue problems. However, there has so far not been any comprehensive survey of differential equations with non-standard growth including and comparing results on existence and regularity. It is the purpose of the present article to fill this gap. Although we were obviously forced to make choices about what to include in detail, we have tried to include in the bibliography all works published to date (in international forums), with an indication in the text of where they fit in.

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We start by sketching the development of the field. Initial interest was in function spaces. Variable exponent Lebesgue spaces appeared in the literature for the first time in a 1931 article by Orlicz [6]. However, investigations then focused on spaces with a modular of the form

$$\varrho(f) = \int_\Omega \varphi(|f(x)|)\,\mathrm{d}x,$$

which are now called Orlicz spaces. Notice that since φ does not depend explicitly on x, the case $|f(x)|^{p(x)}$ is not included. From here, the theory took a more abstract turn, to spaces with modulars not given by a specific function, so-called modular spaces. They were first systematically studied by Nakano [7,8]. Later, a more explicit version of these spaces, modular function spaces, was investigated by Polish mathematicians, like Hudzik, Kamińska and Musielak; cf. [9]. Variable exponent Lebesgue spaces on the real line have been independently developed by the Russian researchers Sharapudinov and Tsenov.

In the mid-80s, Zhikov [10] started a new line of investigation that was to become intimately related to the study of variable exponent spaces, namely he considered variational integrals with non-standard growth conditions. Kováčik [11,12] also found a few results in the 80s and 90s, but they appear not to have been very influential. Zhikov's work was continued by Fan from around 1995 [13–15] and by Alkhutov since 1997 [16]. On a separate path, researchers in Italy, e.g. Marcellini [17], studied minimization problems with (p, q)-growth, i.e. minimizing

$$\int_\Omega F(x, |\nabla u|)\,\mathrm{d}x$$

where $z^p \leqslant F(x, z) \leqslant z^q + 1$ for all $z \geqslant 0$. Note that we recover standard growth conditions if $p = q$. As a special case of (p, q)-growth we have $p(x)$-growth: here $F(x, z) \approx z^{p(x)}$ for some bounded function $p\colon \Omega \to (1, \infty)$. Regularity properties of such functionals were extensively studied by Acerbi, Mingione and their collaborators starting at the end of the 90s. Finally, Růžička and his collaborators [18,19] studied equations with non-standard growth in the modeling of so-called electrorheological fluids; see also [20–22]. As another application, Chen, Levine, and Rao [23] suggested a model for image restoration; see also [24–27].

In the remainder of the article we flesh out this sketch of the development of the field but focus mostly on newer developments. The structure of the presentation is as follows. We start by recalling the well-known p-growth differential equation and show its relation to the $p(\cdot)$-growth condition considered here. We divide the main content of the survey into two parts: existence and regularity.

Within the existence part, we first consider the one-dimensional case. Then we move on to $p(\cdot)$ type Laplace equations

$$-\Delta_{p(\cdot)} u = B(x, u)$$

with increasingly complex right hand side B, namely 0, then a power $|u|^{q(x)-2}u$ of the function, then a general function $f(x, u)$ in Sections 4, 5 and 6, respectively. In the last section of the part we deal with systems of two $p(\cdot)$ type equations.

The regularity theory for equations with non-standard growth seems more complete than the existence theory. In the first section of Part III we deal with harmonic functions and Harnack inequalities. Then we consider elliptic equations and corresponding minimizers, which cover almost all cases of existence from Part II. In the subsequent sections we deal with quasiminimizers, parabolic equations and weaker notions of solutions.

It should be emphasized that we do not aim at giving every detail about the results that we cover. In order to give a clearer picture of the field we have standardized notation and expressed conditions and conclusions of theorems with appropriate modifications from the original sources where necessary. Sometimes, the assumptions that we present are not quite as general as they could be, and at other times we omit certain technical conditions (but we do explicitly state this). Therefore, we strongly suggest that the reader consult the original sources when using the results for further research.

There are a variety of related problems that we could not include in this paper; in particular, problems studied in only a few papers have not been included, since it is possible to get an overview of that research from the cited papers.

- To a minimization problem we can add an additional requirement that our minimizer be larger than a given function. In such *obstacle problems* we minimize, say, the energy

  $$\int_\Omega F(x, u, \nabla u)\,\mathrm{d}x,$$

  or solve some equation, with the additional condition that $u \geqslant \psi$ with ψ given. In the variable exponent setting, obstacle problems have been studied by Eleuteri and Habermann [28–30], Harjulehto, Hästö, Koskenoja, Lukkari and Marola [31], as well as Rodrigues, Sanchón and Urbano [32].
- The standard parabolic *porous medium equation* is

  $$\partial_t u - \operatorname{div}(|u|^\gamma \nabla u) = 0.$$

  If γ is a function, this may be considered a variable exponent problem. Such problems have been studied by Antontsev and Shmarev [33] and Henriques and Urbano [34]. Since the variable exponent acts on $|u|$ and not $|\nabla u|$, the problems are not of variable exponent type with respect to the Sobolev space.