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The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces[☆]

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ABSTRACT

In this paper we prove the existence of solutions of certain kinds of nonlinear fractional integrodifferential equations in Banach spaces. Further, Cauchy problems with nonlocal initial conditions are discussed for the aforementioned fractional integrodifferential equations. At the end, an example is presented.

1. Introduction

The use of fractional differential equations has emerged as a new branch of applied mathematics, which has been used for constructing many mathematical models in science and engineering. In fact fractional differential equations are considered as models alternative to nonlinear differential equations [1] and other kinds of equations [2–4]. The theory of fractional differential equations has been extensively studied by many authors [5–12]. In [13,14] the authors proved the existence of solutions of abstract fractional differential equations by using semigroup theory and the fixed point theorem. Many partial fractional differential or integrodifferential equations can be expressed as fractional differential or integrodifferential equations in some Banach spaces [15].

Byszewski [16] initiated the study of nonlocal Cauchy problems for abstract evolution differential equations. Subsequently several authors discussed the problem for different kinds of nonlinear differential equations and integrodifferential equations including functional differential equations in Banach spaces [17–19]. Balachandran et al. [20–22] established the existence of solutions of quasilinear integrodifferential equations with nonlocal conditions. In these papers the quasilinear operator is unbounded. Recently N'Guerekata [23] and Balachandran and Park [24] investigated the existence of solutions of fractional abstract differential equations with nonlocal initial condition. Benchohra and Seba [25] studied the existence problem for impulsive fractional differential equations in Banach spaces. Balachandran and Kiruthika [26] discussed the nonlocal Cauchy problem with an impulsive condition for semilinear fractional differential equations, whereas Chang and Nieto [27] studied the same problem for neutral integrodifferential equations via fractional

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operators. Belmekki et al. [28] studied the existence of periodic solutions of nonlinear fractional differential equations. Cuevas and Cesar de Souza [29] discussed ω-periodic solutions of fractional integrodifferential equations. In this paper we study the existence of solutions of fractional quasilinear integrodifferential equations in Banach spaces by using the fractional calculus and the Banach fixed point theorem.

2. Preliminaries

We need some basic definitions and properties of fractional calculus which are used in this paper.
Let X be a Banach space and $\mathcal{R}_+ = [0, \infty)$. Suppose $f \in L_1(\mathcal{R}_+)$.

Definition 2.1. The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of function $f \in L_1(\mathcal{R}_+)$ is defined as

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \mathrm{d}s,$$

where $a \in \mathcal{R}$ and $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. The Riemann–Liouville fractional derivative order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathcal{N}$, is defined as

$${}^{R-L}D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s) \mathrm{d}s,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

The Riemann–Liouville fractional derivatives have singularity at zero and the fractional differential equations in the Riemann–Liouville sense require initial conditions at some point different to $x_0 = a$. To overcome this issue, Caputo [30] defined the fractional derivative in the following way.

Definition 2.3. The Caputo fractional derivative order $\alpha > 0$, $n-1 < \alpha < n$, is defined as

$${}^{C}D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^n(s) \mathrm{d}s,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$. If $0 < \alpha < 1$, then

$${}^{C}D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^{\alpha}} \mathrm{d}s,$$

where $f'(s) = Df(s) = \frac{\mathrm{d}f(s)}{\mathrm{d}s}$ and f is an abstract function with values in X.
Now we shall state some properties of the operators I_{a+}^{α} and ${}^{C}D_{a+}^{\alpha}$.

Properties 2.4. *For $\alpha,\ \beta > 0$ and f a suitable function (see for instance [31,32]) we have*

(i) $I_{a+}^{\alpha} I_{a+}^{\beta} f(t) = I_{a+}^{\alpha+\beta} f(t)$;
(ii) $I_{a+}^{\alpha} I_{0}^{\beta} f(t) = I_{a+}^{\beta} I_{a+}^{\alpha} f(t)$;
(iii) $I_{a+}^{\alpha} (f(t) + g(t)) = I_{a+}^{\alpha} f(t) + I_{a+}^{\alpha} g(t)$;
(iv) $I_{a+}^{\alpha}\ {}^{C}D_{a+}^{\alpha} f(t) = f(t) - f(a)$, $0 < \alpha < 1$;
(v) ${}^{C}D_{a+}^{\alpha} I_{a+}^{\alpha} f(t) = f(t)$;
(vi) ${}^{C}D_{a+}^{\alpha} f(t) = I_{a+}^{1-\alpha} Df(t) = I_{a+}^{1-\alpha} f'(t)$, $0 < \alpha < 1$;
(vii) ${}^{C}D_{a+}^{\alpha}\ {}^{C}D_{a+}^{\beta} f(t) \neq {}^{C}D_{a+}^{\alpha+\beta} f(t)$;
(viii) ${}^{C}D_{a+}^{\alpha}\ {}^{C}D_{a+}^{\beta} f(t) \neq {}^{C}D_{a+}^{\beta}\ {}^{C}D_{a+}^{\alpha} f(t)$.

From the above we observe that both the Riemann–Liouville and the Caputo fractional operators do not possess either semigroup or commutative properties, which are inherent to the derivatives of integer order. For basic facts about fractional integrals and fractional derivatives one can refer to the books [31,33,34,32].

Let $C(J, X)$ be the Banach space of continuous functions $x(t)$ with $x(t) \in X$ for $t \in J = [0, T]$ and $\|x\|_{C(J,X)} = \max_{t \in J} \|x(t)\|$. Let $B(X)$ denote the Banach space of bounded linear operators from X into X with the norm $\|A\|_{B(X)} = \sup\{\|A(y)\| : \|y\| = 1\}$. On the other hand, we will write ${}^{C}D_{0+}^{\beta}$ using the notation ${}^{C}D^{\beta}$.

Consider the linear fractional evolution equation

$$\begin{aligned} &{}^{C}D^{q} u(t) = A(t) u(t) + f(t), \quad 0 \le t \le T, \\ &u(0) = u_0, \end{aligned} \tag{1}$$