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A class of polynomial interior-point algorithms for the Cartesian $P_*(\kappa)$ second-order cone linear complementarity problem

G.Q. Wang [a,b,*], D.T. Zhu [a]

[a] Department of Mathematics, Shanghai Normal University, Shanghai 200234, PR China

[b] College of Advanced Vocational Technology, Shanghai University of Engineering Science, Shanghai 200437, PR China

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ABSTRACT

In this paper a class of polynomial interior-point algorithms for the Cartesian $P_*(\kappa)$ second-order cone linear complementarity problem based on a parametric kernel function, with parameters $p \in [0, 1]$ and $q \geq 1$, are presented. The proposed parametric kernel function is used both for determining the search directions and for measuring the distance between the given iterate and the μ-center for the algorithms. Moreover, the currently best known iteration bounds for the large- and small-update methods, namely, $O((1 + 2\kappa)\sqrt{N} \log N \log \frac{N}{\varepsilon})$ and $O((1 + 2\kappa)\sqrt{N} \log \frac{N}{\varepsilon})$, are obtained, respectively, which reduce the gap between the practical behavior of the algorithms and its theoretical performance results.

1. Introduction

In this paper we consider the second-order cone linear complementarity problem (SOCLCP), which seeks vectors $x, s \in \mathbf{R}^n$ and $q \in \mathbf{R}^n$ such that

$$x \in \mathcal{K}, \quad s = \mathcal{A}(x) + q \in \mathcal{K}, \quad \text{and} \quad \langle x, s \rangle = 0,$$

where $\langle x, s \rangle = \mathbf{Tr}(x \circ s)$ denotes the Euclidean inner product, $\mathcal{A} : \mathcal{K} \to \mathcal{K}$ is a linear transformation, and $\mathcal{K} \subseteq \mathbf{R}^n$ is the Cartesian product of several second-order cones, i.e.,

$$\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2 \times \cdots \times \mathcal{K}^N,$$

with

$$\mathcal{K}^j = \left\{(x_1, x_{2:n_j}) \in \mathbf{R} \times \mathbf{R}^{n_j - 1} : x_1 \geq \|x_{2:n_j}\|\right\},$$

for each $j \in J = \{1, 2, \ldots, N\}$, and $n = \sum_{j=1}^{N} n_j$. It includes as a special case the well-known standard linear complementarity problem (SLCP), corresponding to $n_j = 1$ for all j, i.e., $\mathcal{K}$ is the nonnegative orthant $\mathbf{R}^n_+$, and the Karush–Kuhn–Tucker (KKT) optimality conditions for second-order cone optimization (SOCO) [1–3] can be written in the form of SOCLCP [4,5]. Additionally, many important practical problems in economics and engineering, such as facility location and Nash equilibrium [6], can be formulated as it. Many researchers have also studied the second-order cone nonlinear complementarity problem (SOCNCP) and achieved plentiful and beautiful results. For an overview of these results we refer to [7–11].

* Corresponding author at: Department of Mathematics, Shanghai Normal University, Shanghai 200234, PR China.
E-mail addresses: guoq_wang@hotmail.com (G.Q. Wang), dtzhu@shnu.edu.cn (D.T. Zhu).

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We call SOCLCP the Cartesian $P_*(\kappa)$-SOCLCP if the linear transformation $\mathcal{A}$ has the Cartesian $P_*(\kappa)$-property, i.e., for any nonnegative real number κ, the linear transformation $\mathcal{A}$ satisfies

$$(1+4\kappa)\sum_{v\in I_+(x)}\langle x^{(v)},[\mathcal{A}(x)]^{(v)}\rangle+\sum_{v\in I_-(x)}\langle x^{(v)},[\mathcal{A}(x)]^{(v)}\rangle\geq 0, \tag{1}$$

where

$$I_+(x)=\{v\in J:\langle x^{(v)},[\mathcal{A}(x)]^{(v)}\rangle>0\},\quad \text{and}\quad I_-(x)=\{v\in J:\langle x^{(v)},[\mathcal{A}(x)]^{(v)}\rangle<0\}$$

are two index sets. The linear transformation $\mathcal{A}$ has the Cartesian P_*-property if it has the Cartesian $P_*(\kappa)$-property for some nonnegative κ, i.e.,

$$P_*=\bigcup_{\kappa\geq 0}P_*(\kappa).$$

We also recall that the linear transformation $\mathcal{A}$ has the Cartesian P_0-property (respectively, P-property), if for any $x\in\mathcal{K}$ and $x\neq 0$, there exists an index $v\in J$ such that $x^{(v)}\neq 0$ and $\langle x^{(v)},[\mathcal{A}(x)]^{(v)}\rangle\geq 0$ (respectively, $\langle x^{(v)},[\mathcal{A}(x)]^{(v)}\rangle>0$). It is clear that the Cartesian P_* class involves the Cartesian P class and turns out to be a special case in the Cartesian P_0 class [11]. The concept of the Cartesian $P_*(\kappa)$-property was first introduced by Luo and Xiu [12] in the general Euclidean Jordan algebra. Actually, it is a straightforward extension of the $P_*(\kappa)$-matrix introduced by Kojima et al. [13]. The related to the Cartesian P_0- and P-properties which were first introduced by Chen and Qi [14] for a linear transformation between the space of symmetric matrices, and later extended by Pan and Chen [11] to the space of second-order cones. It should be pointed out that the Cartesian $P_*(\kappa)$-property is a weaker property than the monotonicity unless $\kappa=0$. Moreover, the linear transformation $\mathcal{A}$ with the Cartesian $P_*(\kappa)$-property becomes the usual $P_*(\kappa)$ matrix when $\mathcal{K}$ is specified to be $\mathbf{R}^n_+$, correspondingly, the Cartesian $P_*(\kappa)$-SOCLCP reduces to the $P_*(\kappa)$-LCP [13].

Until now all known polynomial interior-point methods (IPMs) used the so-called central path as a guideline to the optimal set, and some variant of Newton's method to follow the central path approximately. Kojima et al. [13] first proved the existence of the central path for any $P_*(\kappa)$-LCP and unified the theory of the $P_*(\kappa)$-LCP from the viewpoint of IPMs. Since then, many polynomial interior-point algorithms have been developed for solving the $P_*(\kappa)$-LCP (see, e.g., [15,16]). However, there is still a gap between the practical behavior of these algorithms and these theoretical performance results with respect to the update strategies of the duality gap parameter in the algorithms. The so-called large-update IPMs have superior practical performance but with relatively weak theoretical results. While the so-called small-update IPMs enjoy the best known worst-case iteration bounds, their performance in computational practice is poor. Recently, Peng et al. [17] presented primal–dual IPMs based on the self-regular proximities for linear optimization (LO) and derived the currently best known iteration bounds for the large- and small-update methods, namely, $O\left(\sqrt{n}\log n\log\frac{n}{\varepsilon}\right)$ and $O\left(\sqrt{n}\log\frac{n}{\varepsilon}\right)$, respectively, which almost closes the gap between the iteration bounds for the large- and small-update methods. Bai et al. proposed a class of primal–dual IPMs for LO [18] based on a variety of non-self-regular kernel functions (i.e., the so-called eligible kernel functions) and obtained the same iteration bounds as in [17]. Later on, Amini and Peyghami [19], Bai et al. [20], Wang and Bai [21], Cho [22], Cho and Kim [23], Cho et al. [24] extended primal–dual interior-point algorithms for LO to $P_*(\kappa)$-LCP based on some eligible kernel functions, respectively.

Motivated by their work, in this paper we propose a class of polynomial interior-point algorithms for the Cartesian $P_*(\kappa)$-SOCLCP based on a parametric kernel function as defined by

$$\psi(t)=\begin{cases}\dfrac{t^{p+1}-1}{p+1}+\dfrac{t^{1-q}-1}{q-1}, & t>0,\ p\in[0,1],\ q>1,\\ \dfrac{t^{p+1}-1}{p+1}-\log t, & t>0,\ p\in[0,1],\ q=1,\end{cases} \tag{2}$$

where p and q are growth and barrier parameters, respectively. This kernel function was first introduced in [25] for LO, which covers a wide range of kernel functions, including the classical logarithmic kernel function, the prototype self-regular functions and also non-self-regular functions. The purpose of the paper is to deal with the Cartesian $P_*(\kappa)$-SOCLCP based on the kernel function $\psi(t)$ uniformly. We adopt the basic analysis used in [25] and revise them to be suited for the Cartesian $P_*(\kappa)$-SOCLCP case. We also develop some new analytic tools that are used in the analysis of the algorithms. Finally, we derive the currently best known iteration bounds for the large- and small-update methods, namely, $O((1+2\kappa)\sqrt{N}\log N\log\frac{N}{\varepsilon})$ and $O((1+2\kappa)\sqrt{N}\log\frac{N}{\varepsilon})$, respectively. Moreover, our analysis is simple and straightforward to the LO analogue.

The paper is organized as follows. In Section 2, we first recall some relevant algebraic properties of the second-order cones. Then we discuss the central path and the new search directions for the Cartesian $P_*(\kappa)$-SOCLCP. The generic polynomial interior-point algorithm for the Cartesian $P_*(\kappa)$-SOCLCP is also presented. In Section 3, we propose some properties of the kernel function $\psi(t)$ and the corresponding barrier function $\Psi(v)$. In Section 4, we analyze the algorithms and derive the currently best known iteration bounds for the large- and small-update methods. Finally, some conclusions and remarks follow in Section 5.

We use the following national conventions. $\mathbf{R}^n$, $\mathbf{R}^n_+$ and $\mathbf{R}^n_{++}$ denote the set of vectors with n components, the set of nonnegative vectors and the set of positive vectors, respectively. $\|.\|$ denotes the Frobenius norm for matrices, and the