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# Asymptotic properties in parabolic problems dominated by a $p$-Laplacian operator with localized large diffusion

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ABSTRACT

This paper is concerned with upper semicontinuity of the family of attractors associated with nonlinear reaction–diffusion equations with principal part governed by a degenerate $p$-Laplacian in which the diffusion $d_\lambda$ blows up in localized regions inside the domain.


## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, be a smooth bounded domain with smooth boundary $\Gamma = \partial\Omega$, and $\lambda \in (0, 1]$ a parameter. In this work we study the asymptotic behavior of the solutions of the family of parabolic equations

$$\begin{cases} u_t^\lambda - \operatorname{div}(d_\lambda(x)|\nabla u^\lambda|^{p-2}\nabla u^\lambda) + |u^\lambda|^{p-2}u^\lambda = B(u^\lambda) & \text{in } \Omega \\ u^\lambda = 0 & \text{on } \Gamma, \\ u^\lambda(0) = u_0^\lambda, \end{cases} \tag{1.1}$$

as $\lambda \to 0$. The parameter $\lambda$ represents the fact that, as $\lambda \to 0$, the diffusion $d_\lambda$ is going to infinity in a localized region $\Omega_0$ inside the physical domain $\Omega$. We assume that $p > 2$ and that $B$ is globally Lipschitz and uniformly integrable.

Next we introduce some notation following [1]. Let $\Omega_0$ be a smooth subdomain of $\Omega$, with $\bar{\Omega}_0 \subset \Omega$, $\Omega_0 = \bigcup_{i=1}^m \Omega_{0,i}$, where $m$ is a positive integer and $\Omega_{0,i}$ are connected smooth subdomains of $\Omega$ with $\bar{\Omega}_{0,i} \cap \bar{\Omega}_{0,j} = \varnothing$, for $i \neq j$. Define $\Omega_1 = \Omega \setminus \bar{\Omega}_0$, and $\Gamma_{0,i} = \partial\Omega_{0,i}$, $\Gamma_0 = \bigcup_{i=1}^m \Gamma_{0,i}$ as the boundaries of $\Omega_{0,i}$ and $\Omega_0$, respectively. Notice that $\partial\Omega_1 = \Gamma \cup \Gamma_0$.

The diffusion coefficients $d_\lambda : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ are bounded and smooth functions in $\Omega$, satisfying

$$0 < m_0 \leqslant d_\lambda(x) \leqslant M_\lambda, \tag{1.2}$$

for all $x \in \Omega$ and $0 < \lambda \leqslant 1$. We also assume that the diffusion is large in $\Omega_0$ as $\lambda \to 0$, or more precisely,

$$d_\lambda(x) \to \begin{cases} d_0(x), & \text{uniformly on } \Omega_1,\ (d_0 \in \mathcal{C}^1(\bar{\Omega}_1, (0, \infty))); \\ \infty, & \text{uniformly on compact subsets of } \Omega_0. \end{cases}$$

It is important to notice here that the assumption that $\Gamma \cap \Gamma_0 = \varnothing$, that is, the diffusion is large in the interior of $\Omega$, is crucial in the development of our analysis.

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If in a reaction–diffusion process the diffusion coefficient behaves as expressed above, intuitively we expect the solutions to tend to become homogeneous in the regions where the diffusion becomes large, that is, for small values of $\lambda$, we expect that the solution of the problem (1.1) will become approximately constant on $\Omega_0$ as occurs in semilinear problems (see [2,3]). For this reason, suppose that $u^\lambda$ converges to $u$ as $\lambda \to 0$, in some sense, and that $u$ takes, on $\Omega_0$, a time dependent spatially constant value, which we will denote by $u_{\Omega_0}(t)$.

Next we intend to obtain the equation that describes the limiting problem. Notice that, since the limit function $u$ is in $W^{1,p}(\Omega)$, its constant value in $\Omega_0$, $u_{\Omega_0}(t)$, cannot be arbitrary. Also, in the boundary $\Gamma_0 = \partial\Omega_0$, we must have $u_{|\Gamma_0} = u_{\Omega_0}(t)$. In $\Omega_1$, we have

$$u_t^\lambda - \operatorname{div}(d_\lambda(x)|\nabla u^\lambda|^{p-2}\nabla u^\lambda) + |u^\lambda|^{p-2}u^\lambda = B(u^\lambda).$$

From properties of convergence of the function $d_\lambda(x)$ in $\Omega_1$ it seems reasonable to have in the limit

$$u_t - \operatorname{div}(d_0(x)|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = B(u), \quad \text{for } u \in W^{1,p}(\Omega).$$

In $\Omega_0$, we have

$$\int_{\Omega_0} u_t^\lambda(x,t)\,dx - \int_{\Omega_0} \operatorname{div}(d_\lambda(x)|\nabla u^\lambda|^{p-2}\nabla u^\lambda)\,dx + \int_{\Omega_0} |u^\lambda|^{p-2}u^\lambda\,dx = \int_{\Omega_0} B(u^\lambda(x,t))\,dx.$$

From Gauss's Divergence Theorem it follows that

$$\int_{\Omega_0} \operatorname{div}(d_\lambda(x)|\nabla u^\lambda|^{p-2}\nabla u^\lambda)\,dx = -\int_{\Gamma_0} d_\lambda(x)|\nabla u^\lambda|^{p-2}\frac{\partial u^\lambda}{\partial \vec{n}}\,dx,$$

where $\vec{n}$ denotes the unit inward normal to $\Omega_0$ in the surface integral and then

$$\int_{\Omega_0} u_t^\lambda(x,t)\,dx + \int_{\Gamma_0} d_\lambda(x)|\nabla u^\lambda|^{p-2}\frac{\partial u^\lambda}{\partial \vec{n}}\,dx + \int_{\Omega_0} |u^\lambda|^{p-2}u^\lambda\,dx = \int_{\Omega_0} B(u^\lambda(x,t))\,dx.$$

Taking the limit as $\lambda \to 0$, we obtain

$$\dot{u}_{\Omega_0}(t)|\Omega_0| + \int_{\Gamma_0} d_0(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \vec{n}}\,dx + \int_{\Omega_0} |u_{\Omega_0}(t)|^{p-2}u_{\Omega_0}(t)\,dx = |\Omega_0|B(u_{\Omega_0}(t)).$$

Dividing both sides by $|\Omega_0|$, we get the following ordinary differential equation:

$$\dot{u}_{\Omega_0}(t) + \frac{1}{|\Omega_0|}\left(\int_{\Gamma_0} d_0(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \vec{n}}\,dx + \int_{\Omega_0} |u_{\Omega_0}(t)|^{p-2}u_{\Omega_0}(t)\,dx\right) = B(u_{\Omega_0}(t)).$$

With these heuristic considerations and assuming that in the limit we will work with a space of constant functions on $\Omega_0$, we can write the limiting problem in the following way:

$$\begin{cases} u_t - \operatorname{div}(d_0(x)|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = B(u) & \text{in } \Omega_1 \\ u_{|\Omega_{0,i}} := u_{\Omega_{0,i}} & \text{in } \Omega_{0,i}, \\ \dot{u}_{\Omega_{0,i}} + \dfrac{1}{|\Omega_{0,i}|}\left[\displaystyle\int_{\Gamma_{0,i}} d_0(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \vec{n}}\,dx + \int_{\Omega_{0,i}} |u_{\Omega_{0,i}}|^{p-2}u_{\Omega_{0,i}}\,dx\right] = B(u_{\Omega_{0,i}}) & \\ u = 0 & \text{on } \Gamma \\ u(0) = u_0, & \end{cases} \tag{1.3}$$

which is a singular limit problem due to the variation of the parameter $\lambda$.

The study of upper semicontinuity of attractors for localized large diffusion semilinear problems has been considered in [4] and lower semicontinuity has been proved in [5]. Attractors for parabolic problems governed by $p$-Laplacian operators, when $p > 2$, have been appearing in the literature since the nineties, and these systems usually present behavior similar to that of the Laplacian problems, despite the results having to be, in general, proved using different tools, avoiding linear arguments [6,7]. The existence of global solutions and a great number of useful properties enjoyed by this kind of problem can be obtained through the theory for monotone operators [8], and one of the first obstacles to studying this class of quasilinear equations was the lack of uniqueness when considering non-globally Lipschitz perturbations of the $p$-Laplacian. At that time, when the parabolic $p$-Laplacian first started to be considered, it was not clear how to deal with multivalued dynamical systems but, from then on, a great understanding of these problems was achieved and, from the combination of several isolated efforts, a well organized theory has been arising [9,10].

The first work considering large diffusion for $p$-Laplacian problems is [11], where it is proved that there exists a positive time from which the spatial gradients of solutions go to zero as the diffusion goes to infinity and, as a simple consequence of the Poincaré–Wirtinger Inequality, all the relevant elements for describing the asymptotic behavior are around their own spatial average if the diffusion is large enough. It is also proved that the attractors continuously approach the attractor of an ordinary equation as the diffusion increases to infinity in the whole domain, but the authors only deal with globally Lipschitz