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Pullback attractors for a non-autonomous generalized 2D parabolic system in an unbounded domain

Jong Yeoul Park, Sun Hye Park *

Department of Mathematics, Pusan National University, Busan 609-735, South Korea

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ABSTRACT

The existence of a pullback attractor is proven for a non-autonomous generalized 2D parabolic system in an unbounded domain. The asymptotic compactness of the solution operator is obtained by the uniform estimates on the tails of solutions.

1. Introduction

In this paper, we study the dynamical behavior of a non-autonomous generalized 2D parabolic system in an unbounded domain. Let Ω_0 be a bounded open subset in $\mathbb{R}$, $\Omega = \Omega_0 \times \mathbb{R}$ with boundary $\partial\Omega$. Consider a non-autonomous generalized 2D parabolic system

$$\begin{aligned}
&-\Delta u_t + \alpha^2 \Delta^2 u_t + \nu \Delta^2 u + \nabla \cdot \overrightarrow{F}(u) + B(u,u) = g(x,t) \quad \text{in } \Omega \times [\tau, \infty),\\
&u = \nabla u = 0 \quad \text{on } \partial\Omega \times [\tau, \infty),\\
&u(x,\tau) = u_\tau(x) \quad \text{in } \Omega,
\end{aligned} \tag{1.1}$$

where $u_t = \frac{\partial u}{\partial t}$, α, ν are positive constants, $\overrightarrow{F}$ is a nonlinear vector function, g is an external forcing term with $g \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$ and $B(u,u) = \frac{\partial u}{\partial x_2}\frac{\partial \Delta u}{\partial x_1} - \frac{\partial u}{\partial x_1}\frac{\partial \Delta u}{\partial x_2}$. If $\overrightarrow{F} \equiv 0$ in (1.1), the system is 2D Navier–Stokes–Voight equation. Navier–Stokes-Voight equation was introduced by Oskolkov [1] to describe a Kelvin–Voight viscoelastic incompressible fluid. Many authors have treated the autonomous Navier–Stokes–Voight equation in bounded domains [2,3] and in unbounded domains [4] from various points of view. When the domain is unbounded, the Sobolev embedding is no longer compact. This gives a difficulty for proving the existence of a global attractor. For some PDEs, such difficulty can be overcome by the energy approach, which is introduced by Ball [5,6]. Polat [7] established the existence of a global attractor to the autonomous problem (1.1), that is, when g is independent of time t, in the unbounded domain by using the technique

* Corresponding author.
E-mail addresses: jyepark@pusan.ac.kr (J.Y. Park), sh-park@pusan.ac.kr (S.H. Park).

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of uniform estimates on the tails of solutions. This technique was developed by Wang [8] to investigate the behavior of reaction–diffusion equations in unbounded domains. In this paper, we intend to investigate the dynamical behavior to the non-autonomous system (1.1) by allowing the external force g to depend on time t.

Non-autonomous equations are also of great importance and interest as they appear in many applications in the natural science. The long-time behavior of solutions of such equations have been studied by many researchers [9–14]. Wang [13] showed the existence of a pullback global attractor for the non-autonomous reaction–diffusion equations on the whole domain $\mathbb{R}^n$ by using the uniform a priori estimates on the tails of solutions. Motivated by the ideas in [7,13,15], we study the existence of the pullback attractor for the non-autonomous system (1.1) on unbounded domains. The plan of this paper is as follows. In Section 2, we give some basic definitions and abstract results concerning the pullback attractor for non-autonomous dynamical systems. In Section 3, we derive uniform estimates of solutions by employing the techniques in [13,15]. Finally, we prove the existence of a pullback attractor for the system (1.1).

2. Preliminaries and abstract results

In this section, we give some basic definitions and abstract results concerning the pullback attractor for non-autonomous dynamical systems. These definitions and results can be found in [9,16,17,13] and references therein.

Definition 2.1. A family of mappings $\{\theta_t\}_{t\in\mathbb{R}}$ from Q to itself is called a family of shift operators on Q, if $\{\theta_t\}_{t\in\mathbb{R}}$ satisfies the following group properties:

(i) $\theta_0(q) = q$ for all $q \in Q$,
(ii) $\theta_{t+\tau}(q) = \theta_t(\theta_\tau(q))$ for all $q \in Q$ and $t, \tau \in \mathbb{R}$.

Definition 2.2. Let $\{\theta_t\}_{t\in\mathbb{R}}$ be a family of shift operators on Q. Then a continuous θ-cocycle ϕ on X is a mapping $\phi : \mathbb{R}^+ \times Q \times X \to X$ satisfying, for all $q \in Q$ and $t, \tau \in \mathbb{R}$,

(i) $\phi(0, q, \cdot)$ is the identity on X;
(ii) $\phi(t+\tau, q, \cdot) = \phi(t, \theta_\tau(q), \phi(\tau, q, \cdot))$;
(iii) $\phi(t, q, \cdot) : X \to X$ is continuous.

Suppose $\mathcal{D}$ is a nonempty class of parameterized sets $D = \{D(q)\}_{q\in Q}$, $D(q) \subset X$ for every $q \in Q$.

Definition 2.3. It is said that $B = \{B(q)\}_{q\in Q} \in \mathcal{D}$ is pullback $\mathcal{D}$-absorbing for ϕ in $\mathcal{D}$ if for every $q \in Q$ and $D \in \mathcal{D}$, there exists $T(q, D) > 0$ such that

$$\phi(t, \theta_{-t}(q), D(\theta_{-t}(q))) \subset B(q), \quad \forall t \geq T(q, D).$$

Definition 2.4. The cocycle ϕ is said to be pullback $\mathcal{D}$-asymptotically compact in X, if for every $q \in Q$, $\{\phi(t_n, \theta_{-t_n}(q), x_n)\}$ has a convergent subsequence in X whenever $t_n \to +\infty$ and $x_n \in D(\theta_{-t_n}(q))$ with $\{D(q)\}_{q\in Q} \in \mathcal{D}$.

Definition 2.5. A family $C = \{C(q)\}_{q\in Q} \in \mathcal{D}$ is said to be pullback $\mathcal{D}$-attracting if

$$\lim_{t\to\infty} \text{dist}(\phi(t, \theta_{-t}(q), D(\theta_{-t}(q))), C(q)) = 0 \quad \text{for all } q \in Q,\ D \in \mathcal{D},$$

where $\text{dist}(X, Y) = \sup_{x\in X} \inf_{y\in Y} d(x, y)$ is the Hausdorff semi-distance between X and Y.

Definition 2.6. It is said that $A = \{A(q)\}_{q\in Q} \in \mathcal{D}$ is a pullback $\mathcal{D}$-attractor if it satisfies

(1) $A(q)$ is compact in X for any $q \in Q$;
(2) A is pullback $\mathcal{D}$-attracting;
(3) A is invariant, that is, $\phi(t, q, A(q)) = A(\theta_t(q))$ for all $(t, q) \in \mathbb{R}_+ \times Q$.

Theorem 2.1. *Let ϕ be a continuous θ-cocycle on X and there exists a family of pullback $\mathcal{D}$-absorbing sets $\{B(q)\}_{q\in Q} \in \mathcal{D}$. If ϕ is a pullback $\mathcal{D}$-asymptotically compact in X, then ϕ has a unique pullback $\mathcal{D}$-global attractor $\{A(q)\}_{q\in Q} \in \mathcal{D}$ defined by*

$$A(q) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \phi(t, \theta_{-t}(q), B(\theta_{-t}(q)))}.$$

3. Uniform estimates of solutions

Throughout this paper, we denote by $L^p(\Omega)$, $1 \leq p \leq \infty$, and $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$ the usual Lebesgue and Sobolev spaces, respectively. $(\cdot, \cdot)$ denotes the inner product of $L^2(\Omega)$ and $\|\cdot\|$ the induced norm. For a Banach space X, $\|\cdot\|_X$