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Existence of positive solutions for first order discrete periodic boundary value problems with delay

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ABSTRACT

Let $T > 3$ be a positive integer and $\mathbb{T} = \{0, 1, 2, \ldots, T-1\}$. We prove the existence of positive periodic solutions of the nonlinear discrete boundary value problem

$$\Delta u(t) = a(t)g(u(t))u(t) - \lambda b(t)f(u(t-\tau(t))), \quad t \in \mathbb{T},$$
$$u(0) = u(T).$$

Our approach is based upon the global bifurcation techniques.

1. Introduction

Let $T > 3$ be a positive integer and $\mathbb{T} = \{0, 1, 2, \ldots, T-1\}$. Let $\mathbb{R}$ denote the real number set, $\mathbb{Z}$ the integer set. In this paper, we investigate the existence of positive periodic solutions for the following discrete periodic boundary value problems with delay

$$\begin{aligned} &\Delta u(t) = a(t)g(u(t))u(t) - \lambda b(t)f(u(t-\tau(t))), \quad t \in \mathbb{T}, \\ &u(0) = u(T), \end{aligned} \tag{1.1}$$

where $a, b : \mathbb{Z} \to [0, \infty)$ are T-periodic functions, $\tau : \mathbb{Z} \to \mathbb{Z}$ is a T-periodic function. $f, g \in C([0, \infty), [0, \infty))$, $\lambda > 0$ is a parameter.

In recent years, there has been considerable interest in the existence of positive solutions of the following equation:

$$x'(t) = \tilde{a}(t)\tilde{g}(x(t))x(t) - \lambda \tilde{b}(t)\tilde{f}(x(t-\tau(t))), \tag{1.2}$$

where $\tilde{a}, \tilde{b} \in C(\mathbb{R}, [0, \infty))$ are ω-periodic functions, $\int_0^\omega \tilde{a}(t)\mathrm{d}t > 0$, $\int_0^\omega \tilde{b}(t)\mathrm{d}t > 0$ and τ is a continuous ω-periodic function. (1.2) has been proposed as a model for a variety of physiological processes and conditions including production of blood cells, respiration, and cardiac arrhythmias. See, for example, [1–12] and the references therein.

So far, relatively little is known about the existence of positive periodic solutions of (1.1). To the best of our knowledge, only Raffoul [13] dealt with the special equations of (1.1) of the form

$$\Delta x(t) = \alpha(t)x(t) - \lambda b(t)f(x(t-\tau(t))), \tag{1.3}$$

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with $\alpha(t) = a(t) - 1 > 0$, and determined values of λ, for which there exist T-periodic positive solutions of (1.3). Zhou [14] proved the existence results of positive periodic solutions of first order difference equations (with no delay) by using Krasnosel'skii's fixed point theorem.

However, the conditions used in [13] are not sharp, and the main results in [13] give not any information about the global structure of the set of positive periodic solutions.

It is the purpose of this paper to study the global structure of the set of positive periodic solutions of (1.1), and accordingly, to establish some existence results of the positive periodic solutions of (1.1). Our results are sharp. The main tool we used is the Dancer global bifurcation theorem (see [15, Corollary 15.2]).

To obtain it, we make the assumptions:

(H1) $a, b : \mathbb{Z} \to [0, +\infty)$ are T-periodic functions, and $a(t) \not\equiv 0$, $b(t) \not\equiv 0$ on $t \in \mathbb{T}$, $\tau : \mathbb{Z} \to \mathbb{Z}$ is a T-periodic function;

(H2) $f, g : [0, +\infty) \to [0, +\infty)$ are continuous, $0 < l \le g(u) \le L < \infty$ with l, L are positive constants, and $f(u) > 0$, for $u > 0$. Also,

$$\sigma_l = \prod_{s=0}^{T-1}(1 + a(s)l), \qquad \sigma_L = \prod_{s=0}^{T-1}(1 + a(s)L), \tag{1.4}$$

it is clear that $1 < \sigma_l \le \sigma_L$. Let

$$M(r) = \max_{0\le x\le r}\{f(x)\}, \qquad m(r) = \min_{\frac{\sigma_l - 1}{(\sigma_L - 1)\sigma_L}\cdot r \le x \le r}\{f(x)\}. \tag{1.5}$$

The rest of this paper is organized as follows. In Section 2, we give some notations and the main results. Section 3 is devoted to prove the main results, and finally, we give an example to illustrate our main result.

2. Notations and the main results

Recall $\mathbb{T} = \{0, 1, 2, \ldots, T - 1\}$ and let $\hat{\mathbb{T}} = \{0, 1, 2, \ldots, T\}$.

Let

$$E = \{u : \hat{\mathbb{T}} \to \mathbb{R} | u(0) = u(T)\}$$

with the norm $\|u\|_E = \max_{t\in\hat{\mathbb{T}}} |u(t)|$. Then $(E, \|\cdot\|_E)$ is a Banach space.

Let

$$Y = \{u | u : \mathbb{T} \to \mathbb{R}\}$$

with the norm $\|u\|_Y = \max_{t\in\mathbb{T}} |u(t)|$. Then $(Y, \|\cdot\|_Y)$ is a Banach space.

It is easy to see that the operator $\chi : E \to Y$

$$\chi(u(0), u(1), \ldots, u(T - 1), u(T)) = (u(0), u(1), \ldots, u(T - 1))$$

is a homomorphism.

By a *positive solution* of (1.1) we mean a pair (λ, u), where $\lambda > 0$ and u is a solution of (1.1) with $u > 0$ in $\hat{\mathbb{T}}$.

Let $\Sigma \subset \mathbb{R}^+ \times E$ be the closure of the set of positive solutions of (1.1).

We extend the function f to a continuous function $\tilde{f}$ defined on $\mathbb{R}$ in such a way that $\tilde{f} > 0$ for all $s < 0$. For $\lambda > 0$, we then look at arbitrary solutions u of the eigenvalue problem

$$\begin{aligned} &\Delta u(t) = a(t)g(u(t))u(t) - \lambda b(t)\tilde{f}(u(t - \tau(t))), \quad t \in \mathbb{T}, \\ &u(0) = u(T). \end{aligned} \tag{2.1}$$

It is well known that in [16, Theorem 1.1] that (2.1) is equivalent to

$$u(t) = \lambda \sum_{s=0}^{T-1} G_u(t, s)b(s)f(u(s - \tau(s)))\mathrm{d}s,$$

where

$$G_u(t, s) = \frac{\prod_{l=s+1}^{T-1}(1 + a(s)g(u(s)))}{\prod_{s=0}^{T-1}(1 + a(s)g(u(s))) - 1}, \quad s \in \mathbb{T}.$$

Notice that $a(t) \not\equiv 0$ for all $t \in \mathbb{T}$, $0 < l \le g(u) \le L < \infty$, we have

$$\frac{1}{\sigma_L - 1} \le G_u(t, s) \le \frac{\sigma_L}{\sigma_l - 1}, \quad s \in \mathbb{T},$$