Nonlinear Analysis 74 (2011) 4234–4240

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On completeness of quadratic systems

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ARTICLE INFO

Article history:
Received 25 September 2010
Accepted 29 March 2011
Communicated by Ravi Agarwal

MSC:
37C10

Keywords:
Quadratic systems
Polynomial systems
Lorenz system
Autonomous differential equations
Asymptotic behavior
Compactification
Completeness of dynamical system
Invariant set

ABSTRACT

A dynamical system is called complete if every solution of it exists for all $t \in \mathbb{R}$. Let K be the dimension of the vector space of quadratic systems. The set of complete quadratic systems is shown to contain a vector subspace of dimension $2K/3$. We provide two proofs, one by the Gronwall lemma and the second by compactification that is capable of showing incompleteness as well. Characterization of a vector subspace of complete quadratic systems is provided. The celebrated Lorenz system for all real ranges of its parameters is shown to belong to this subspace. We also provide a sufficient condition for a system to be incomplete.

1. Introduction

Among the polynomial systems, quadratic systems have attracted a great deal of attention owing to their important role in mathematical sciences. These systems model diverse natural phenomena, from fluid mechanics to stars' constellations [1–17]. They share similar features with the competing species model and the Lotka–Volterra system; see e.g., [6,9–12,18,15,16]. About eight hundred papers on quadratic systems are mentioned in [19]. The Lorenz system has been a celebrated quadratic system; see the original work of [13] and compare with [20–22]. It continues in recent years to be a source for simulation and generalizations.

It would be beneficial to modeling if we would have criteria that will inform us which dynamical systems are complete and which are not. It would be very helpful to know which systems possess global solutions for all parameters involved for all time and for all initial values. This is so for more reasons than one. On the one hand, a system with solutions that blow up in finite time could be indicative of a break-down of a model. On the other hand, modeling certain phenomena by families of differential systems that are known in advance to possess solutions that exist for all time has an obvious advantage. It goes without saying that criteria for systems that are not complete are of great importance as well.

The purpose of this study is to determine in a sense to be made precise, how large is the subspace of complete quadratic systems and to some extent to characterize this subspace. To this end, we proceed with some preliminary definitions and notations.

Let y be a column vector in $\mathbb{R}^k$, and let $y^\dagger = (y_1, y_2, \dots, y_k)$ denote the row vector transpose of y. Let $f(y) := (f(y)_1, f(y)_2, \dots, f(y)_k)^\dagger$ be a vector field in $\mathbb{R}^k$ where each $f(y)_j$ is a real-valued polynomial function. We say that $\dot{y} =$

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0362-546X/$ – see front matter
doi:10.1016/j.na.2011.03.061

$\frac{dy}{dt} = f(y)$ is a polynomial system of degree L if the vector function $f(y)$ is given by

$$f(y) = f_0(y) + f_1(y) + \cdots + f_{L-1}(y) + f_L(y), \tag{1.1}$$

where each $f_i(y) = (f_i(y)_1, f_i(y)_2, \ldots, f_i(y)_k)^\dagger$ is a column vector of homogeneous polynomials of degree i, for $i = 0, 1, 2, \ldots, L$, and $f_L(y) \neq 0$ for some $y \in \mathbb{R}^k$.

Definition 1. A system of differential equations $\dot{y} = f(y)$ (or the associated vector field $f(y)$) is called *complete* if every solution to the system exists *for all* $t \in \mathbb{R}$.

By a "Lorenz system" we mean a system satisfying

$$\begin{aligned} \dot{y}_1 &= \sigma(y_2 - y_1) \\ \dot{y}_2 &= \rho y_1 - y_2 - y_1 y_3 \\ \dot{y}_3 &= -\beta y_3 + y_1 y_2 \end{aligned} \tag{1.2}$$

for any real values of the parameters.

Let $\mathcal{L} = \mathcal{L}(\mathbb{R}^k)$ be the set of quadratic systems

$$\dot{y} = f_2(y) + f_1(y) + f_0, \quad \text{with } y^\dagger f_2(y) \equiv 0. \tag{1.3}$$

We call the autonomous system (1.3) a Lorenz-like system, because it generalizes certain features of the classical Lorenz system. It is easy to see that all Lorenz systems are in $\mathcal{L}(\mathbb{R}^3)$.

One of the previously unanswered questions in the literature is whether or not the Lorenz system is complete for *all* real values of the parameters.[1] We fill this gap and answer this question in the affirmative for the much larger family of systems, to be named $\mathcal{NAL}$. We have not found in the literature a proof of this global existence for the Lorenz system without restricting the parameters (σ, ρ, β) to be positive, but our proof works for the much larger class. The earliest known proof for the Lorenz system (1.2) is by means of a Lyapunov function, which is not readily available for most Lorenz-like systems $\mathcal{L}$. We chose as a second method of proof the compactification [24,25] that is able to answer not only questions of completeness but is also able to address issues of incompleteness as well.

In Section 2, we prove that the dimension of the subspace of complete quadratic systems is 2/3 of the dimension of the entire space of quadratic systems. We pinpoint a functional cause for completeness. In Section 3, we characterize some incomplete systems and give a condition which guarantees incompleteness. Section 4 is dedicated to remarks and comparisons with some of the related voluminous literature.

2. Completeness and structure of Lorenz-like systems

In this section we prove the main theorem that shows as a corollary that the Lorenz system is complete for all its real parameters. This completeness property is shared by a larger family of non-autonomous quadratic systems that is denoted below by $\mathcal{NAL}$.

Definition 2. Let $CB(\mathbb{R})$ be the family of scalar functions continuous and bounded on $\mathbb{R}$. Let $f_2(t, y)$ be a column vector in $\mathbb{R}^k$ whose components are quadratic forms: $f_2(t, y)_n = (y^\dagger f_{2n}(t) y)$, with each $f_{2n}(t)$ a lower triangular matrix with entries in $CB(\mathbb{R})$. Let $f_1(t, y) = f_1(t)y$, where $f_1(t)$ is a $k \times k$ matrix with entries in $CB(\mathbb{R})$, and let $f_0 = f_0(t)$ be a column vector in $\mathbb{R}^k$ with entries in $CB(\mathbb{R})$. Then $\mathcal{NAL}$ (Non-Autonomous Lorenz-like) is the class of systems

$$\dot{y} = f_2(t, y) + f_1(t, y) + f_0(t), \quad \text{with } y^\dagger f_2(t, y) = 0. \tag{2.1}$$

The completeness of $\mathcal{NAL}$ is given in the theorem below. It also includes a more detailed description of the structure of $\mathcal{L}$ that could explain the orthogonality property in (1.3) as a source of the completeness.

Let $\mathcal{N}$ be the linear space of all (at most) quadratic systems on $\mathbb{R}^k$. The main result of our study is the following.

Theorem 3. (i) *All systems in* $\mathcal{NAL}$ *are complete.* (ii) $\dim(\mathcal{L}) = \frac{k}{3}(k+1)(k+2) = \frac{2}{3}\dim(\mathcal{N})$ *and for systems in* $\mathcal{L}$ *the elements* f_{2n}^{ij} *of the lower triangular matrix* f_{2n} *satisfy the following relations:*

$$f_{2n}^{nn} = 0, \quad \text{for } n = 1, 2, \ldots, k, \tag{2.2}$$

$$\text{for } j \neq n, \quad f_{2j}^{nn} + f_{2n}^{jn} + f_{2j}^{nj} = 0 \tag{2.3}$$

$$\text{for } j < i < n, \quad f_{2n}^{ij} + f_{2i}^{nj} + f_{2j}^{ni} = 0. \tag{2.4}$$

Note that in the second equation, either the second or the third term is 0 *because the matrix is triangular.*

[1] A proof that the Lorenz system with positive parameters is "complete in forward time" (that is, that all solutions exist for all time $t > 0$) can be found in [21] using a Lyapunov function. A proof that it is "complete in backward time" (using the same Lyapunov function) due to Meisters [23] may also be found in [20].