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Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces

Bessem Samet [a,*], Calogero Vetro [b]

[a] Ecole Supérieure des Sciences et Techniques de Tunis, Département de Mathématiques, 5, Avenue Taha Hussein-tunis, B.P.: 56, Bab Menara-1008, Tunisie
[b] Dipartimento di Matematica e Informatica, University of Palermo, Via Archirafi 34, 90123 Palermo, Italy

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ABSTRACT

In this paper, we establish two coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces. The theorems presented extend some results due to Ćirić (2009) [3]. An example is given to illustrate the usability of our results.

1. Introduction

Let (X, d) be a metric space. We denote by $CB(X)$ the collection of non-empty closed bounded subsets of X. For $A, B \in CB(X)$, and $x \in X$, suppose that

$$D(x, A) = \inf_{a \in A} d(x, a)$$

and

$$H(A, B) = \max\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\}.$$

Such a mapping H is called a Hausdorff metric in $CB(X)$ induced by d.

Definition 1.1. An element $x \in X$ is said to be a fixed point of a set-valued mapping $T : X \to CB(X)$ if and only if $x \in Tx$.

The existence of fixed points for various multi-valued contractive mappings has been studied by many authors under different conditions. For details, we refer the reader to [1–10] and the references therein. In 1969, Nadler [11] extended the famous Banach Contraction Principle [12] from single-valued mapping to multi-valued mapping and proved the following fixed point theorem for the multi-valued contraction.

* Corresponding author.
E-mail addresses: bessem.samet@gmail.com (B. Samet), cvetro@math.unipa.it (C. Vetro).

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Theorem 1.1 (*See Nadler [11]*). *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $c \in [0, 1)$ such that*

$$H(Tx, Ty) \leq cd(x, y)$$

for all $x, y \in X$. Then, T has a fixed point.

In 1989, Mizoguchi and Takahashi [8] proved the following interesting fixed point theorem for a weak contraction.

Theorem 1.2 (*See Mizoguchi and Takahashi [8]*). *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for all $x, y \in X$, where α is a function from $[0, +\infty)$ into $[0, 1)$ satisfying the condition $\limsup_{s\to t^+} \alpha(s) < 1$ for all $t \in [0, +\infty)$. Then, T has a fixed point.

In fact, the Mizoguchi–Takahashi fixed point theorem is a generalization of Nadler's fixed point theorem, but its primitive proof is different.

Let $CL(X) := \{A \subseteq X \mid A \neq \emptyset, \overline{A} = A\}$, where $\overline{A}$ denotes the closure of A in the metric space (X, d). In this context, Ćirić [3] proved the following interesting theorem.

Theorem 1.3 (*See Ćirić [3]*). *Let (X, d) be a complete metric space and let T be a mapping from X into $CL(X)$. Let $f : X \to \mathbb{R}$ be the function defined by $f(x) = d(x, Tx)$ for all $x \in X$. Suppose that f is lower semi-continuous and that there exists a function $\varphi : [0, +\infty) \to [a, 1)$, $0 < a < 1$, satisfying*

$$\limsup_{r\to t^+} \varphi(r) < 1 \quad \text{for each } t \in [0, +\infty). \tag{1}$$

Assume that for any $x \in X$ there is $y \in Tx$ satisfying the following two conditions:

$$\sqrt{\varphi(f(x))}d(x, y) \leq f(x) \tag{2}$$

such that

$$f(y) \leq \varphi(f(x))d(x, y). \tag{3}$$

Then, there exists $z \in X$ such that $z \in Tz$.

Recently, there have been very many exciting developments in the field of existence of fixed points in partially ordered sets. For details, we refer the reader to [13–33] and the references therein.

In 2006, Bhaskar and Lakshmikantham [15] introduced the following notion of a coupled fixed point.

Definition 1.2 (*See Bhaskar and Lakshmikantham [15]*). Let X be a non-empty set and $F : X \times X \to X$ be a given mapping. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping F if $F(x, y) = x$ and $F(y, x) = y$.

In [15], Bhaskar and Lakshmikantham established some coupled fixed point theorems for partially ordered metric spaces. They noted that their theorems can be used to investigate a large class of problems and have discussed the existence and uniqueness of a solution for a periodic boundary value problem. Lakshmikantham and Ćirić [21] proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces which extend the results of Bhaskar and Lakshmikantham [15]. For more details on coupled fixed point theory, we also refer the reader to [13,14,20,30–32] and the references therein.

In this paper, we prove two coupled fixed point theorems by considering metric spaces endowed with partial order. Our results give a significant extension of some recent results due to Ćirić [3]. An example is also given to illustrate the usability of our results.

2. The main results

Let (X, d) be a metric space endowed with a partial order $\preceq$. We recall the following definition.

Definition 2.1. A function $f : X \times X \to \mathbb{R}$ is called lower semi-continuous if and only if for any $\{x_n\} \subset X$, $\{y_n\} \subset X$ and $(x, y) \in X \times X$, we have

$$\lim_{n\to+\infty} (x_n, y_n) = (x, y) \Rightarrow f(x, y) \leq \liminf_{n\to+\infty} f(x_n, y_n).$$

Let $G : X \to X$ be a given mapping. We define the set $\Delta \subseteq X \times X$ by

$$\Delta := \{(x, y) \in X \times X \mid G(x) \preceq G(y)\}.$$