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Bifurcation of infinite Prandtl number rotating convection

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ABSTRACT

We consider infinite Prandtl number convection with rotation which is the basic model in geophysical fluid dynamics. For the rotation free case, the rigorous analysis has been provided by Park (2005, 2007, revised for publication) [5,6,25] under various boundary conditions. By thoroughly investigating we prove in this paper that the solutions bifurcate from the trivial solution $u = 0$ to an attractor Σ_R which consists of only one cycle of steady state solutions and is homeomorphic to S^1. We also see how intensively the rotation inhibits the onset of convective motion. This bifurcation analysis is based on a new notion of bifurcation, called attractor bifurcation which was developed by Ma and Wang (2005); see [15].

1. Introduction

The effect of rotation on the stability problem of a fluid layer heated from below has been regarded as an important phenomenon in Rayleigh–Bénard convection. Typically, the motion is described by Boussinesq equations (see, for example, [1,2]). It is not surprising that many papers have been published regrading this basic convection problem because it has geophysical and astrophysical relevance.

In this paper we explore the structure of bifurcated solutions of the rotating Bénard convection. Rayleigh–Bénard convection, that is, a buoyancy-driven convection in a fluid layer heated from below and cooled from above, is one of the prime examples of bifurcating high-dimensional systems. It has long been a subject of intense theoretical and experimental study and has been applied to many different areas of study such as meteorology, geophysics, and astrophysics. In rotating convection, the entire fluid is rotated about a vertical axis with a constant rotation rate Ω.

The governing equations are the following Boussinesq equations:

$$\frac{1}{P_r}\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] + \nabla p - \Delta\mathbf{u} + \sqrt{\mathrm{Ta}}\,\mathbf{k}\times\mathbf{u} - RT\mathbf{k} = 0, \tag{1.1}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T - \Delta T = 0, \tag{1.2}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1.3}$$

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where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity field, p is the pressure function, T is the temperature field and $\mathbf{k} = (0, 0, 1)$ is the unit vector in the x_3-direction.

In the Boussinesq equations, we have three important numbers: the Rayleigh number,

$$R = \frac{g\alpha(T_2 - T_1)h^3}{\nu\kappa},$$

which measures the ratio of overall buoyancy force to the damping coefficients, the Prandtl number,

$$P_r = \frac{\nu}{\kappa},$$

which measures the relative importance of kinematic viscosity over thermal diffusivity, and the Taylor number,

$$\mathrm{Ta} = \frac{4\Omega^2 h^2}{\nu^2}$$

which measures the rotation.

Here, ν and κ are the kinematic viscosity and thermal diffusive coefficients respectively, α is the thermal expansion coefficient of the fluid, g is the gravitational constant, h is the distance between two plates confining the fluid and $T_2 - T_1$ is the temperature difference between the bottom and top plates.

Due to the fact that mathematical information is very limited, the complicated equations have been simplified. For example, the infinite Prandtl number limit of the Boussinesq equations has been used as the standard model for the convection of earth's mantle, where it is argued that P_r could be of the order 10^{24}, as well as for many gasses under high pressure. A Broader rationale for investigating the infinite Prandtl number convection is based on the observation of both the linear and weakly nonlinear theories; that fluids with $P_r > 1$ convect in a similar fashion. Moreover, the infinite Prandtl number model of convection can also be justified as the limit of the Boussinesq approximation to the Rayleigh–Bénard convection as the Prandtl number approaches infinity [3,4].

In the limit of the infinite Prandtl number, the inertial terms in the momentum equation can be dropped, thus we are left with a linear dependence of the velocity field on temperature:

$$\nabla p - \Delta\mathbf{u} + \sqrt{\mathrm{Ta}}\,\mathbf{k} \times \mathbf{u} - RT\mathbf{k} = 0, \tag{1.4}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T - \Delta T = 0, \tag{1.5}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{1.6}$$

The basic linear profiles of (1.4)–(1.6) are steady state solutions given by

$$\mathbf{u} = 0,$$
$$T = 1 - x_3,$$
$$p = p_0 + R\left(x_3 - \frac{x_3^2}{2}\right)\mathbf{k}.$$

Let q be the difference between p and the steady state solutions, and let θ be the difference between T and the steady states solutions, i.e, $p = p_0 + R(x_2 - \frac{x_2^2}{2})k + q$ and $T = (1 - x_2) + \theta$. Then the equations for the perturbation of these trivial solutions are derived as

$$\nabla q - \Delta\mathbf{u} + \sqrt{\mathrm{Ta}}\,\mathbf{k} \times \mathbf{u} - R\theta\mathbf{k} = 0,$$
$$\frac{\partial\theta}{\partial t} + (\mathbf{u} \cdot \nabla)\theta - \Delta\theta - u_3 = 0,$$
$$\nabla \cdot \mathbf{u} = 0.$$

Let $T = \sqrt{R}\theta$ and $p = q$. Then we have

$$\nabla p - \Delta\mathbf{u} + \sqrt{\mathrm{Ta}}\,\mathbf{k} \times \mathbf{u} - \sqrt{R}T\mathbf{k} = 0, \tag{1.7}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T - \Delta T - \sqrt{R}u_3 = 0, \tag{1.8}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{1.9}$$

These rotating infinite Prandtl number system is easier to analyze than the Boussinesq equations since the global existence of smooth solutions can be derived. In particular, by investigating the structure of the attractor generated only by T(much more convenient), it is possible to reconstruct the structure of the attractor in terms of $(\mathbf{u}, T)$, which has the same topological structure as is obtained in terms of only T.

We impose the periodic boundary condition with spatial periods L_1 and L_2 in the horizontal direction x_1 and x_2, respectively: