Nonlinear Analysis 73 (2010) 2078–2087

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Convergence and certain control conditions for hybrid viscosity approximation methods

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ARTICLE INFO

Article history:
Received 12 March 2010
Accepted 12 May 2010

MSC:
47H10
47H09
47H17

Keywords:
Hybrid viscosity approximation method
Nonexpansive mapping
Uniformly smooth Banach space
Hilbert space
Fixed point
Convergence

ABSTRACT

Very recently, Yao, Chen and Yao [20] proposed a hybrid viscosity approximation method, which combines the viscosity approximation method and the Mann iteration method. Under the convergence of one parameter sequence to zero, they derived a strong convergence theorem in a uniformly smooth Banach space. In this paper, under the convergence of no parameter sequence to zero, we prove the strong convergence of the sequence generated by their method to a fixed point of a nonexpansive mapping, which solves a variational inequality. An appropriate example such that all conditions of this result are satisfied and their condition $\beta_n \to 0$ is not satisfied is provided. Furthermore, we also give a weak convergence theorem for their method involving a nonexpansive mapping in a Hilbert space.

1. Introduction

Let X be a real Banach space with the dual space X^*. The value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^* \rangle$. The normalized duality mapping J from X into the family of nonempty (by Hahn–Banach theorem) weak-star compact subsets of X^* is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X.$$

It is known that the norm of X is said to be Gâteaux differentiable (and X is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for every x, y in $U = \{x \in X : \|x\| = 1\}$ the unit sphere of X. The norm of X is said to be uniformly Fréchet differentiable (and X is said to be uniformly smooth) if the limit in (1.1) is attained uniformly for $(x, y) \in U \times U$. Every uniformly smooth Banach space X is reflexive and smooth.

Recall also that if X is smooth then J is single-valued and continuous from the norm topology of X to the weak-star topology of X^*, i.e., norm-to-weak* continuous. It is also well known that if X is uniformly smooth then J is uniformly continuous

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0362-546X/$ – see front matter
doi:10.1016/j.na.2010.05.036

on bounded subsets of X from the strong topology of X to the strong topology of X^*, i.e., uniformly norm-to-norm continuous on any bounded subset of X. See [1,2] for more details.

Let C be a nonempty closed convex subset of X. Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|, \quad \forall x, y \in C.$$

As in [3], we use the notation Π_C to denote the collection of all contractions on C, i.e.,

$$\Pi_C = \{f : C \to C \text{ a contraction}\}.$$

Note that each $f \in \Pi_C$ has a unique fixed point in C.

A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

A point $x \in C$ is a fixed point of T if $Tx = x$. Let $\text{Fix}(T)$ denote the set of fixed points of T; that is, $\text{Fix}(T) = \{x \in C : Tx = x\}$.

If $f : C \to C$ is a contraction, then the Banach Contraction Principle tells us that starting from any fixed element $x \in C$, the iterate $x_{n+1} = f^n(x)$ converges strongly to a unique fixed point of f. However, simple examples show that the above fact is no longer valid for nonexpansive mappings. One approach for solving this complexity is to employ the Mann iteration method which produces a sequence $\{x_n\}$ via the recursive scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \ge 0, \tag{1.2}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily. For convergence results for (1.2) and related iterative schemes, see, e.g., [4–18] and the references therein.

Recently, Xu [3] applied the viscosity approximation method for selecting a particular fixed point of a given nonexpansive mapping and established the strong convergence theorems.

Theorem 1.1 (*[3, Theorem 3.2]*). *Let H be a Hilbert space, C a nonempty closed convex subset of H, $T : C \to C$ a nonexpansive mapping with* $\text{Fix}(T) \ne \emptyset$, *and $f : C \to C$ a contraction. For an arbitrary $x_0 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \quad \forall n \ge 0, \tag{1.3}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies

(H1) $\alpha_n \to 0$;
(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(H3) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then, $x_n \to \bar{x}$, where $\bar{x}$ is the unique solution of the variational inequality

$$\langle (I - f)\bar{x}, x - \bar{x}\rangle \ge 0, \quad \forall x \in \text{Fix}(T).$$

Theorem 1.2 (*[3, Theorem 4.2]*). *Let X be a uniformly smooth Banach space, C a nonempty closed convex subset of X, $T : C \to C$ a nonexpansive mapping with* $\text{Fix}(T) \ne \emptyset$, *and $f \in \Pi_C$. For an arbitrary $x_0 \in C$, let $\{x_n\}$ be generated by* (1.3) *where $\{\alpha_n\} \subset (0, 1)$ satisfies hypotheses* (H1)–(H3) *in* Theorem 1.1. *Then $\{x_n\}$ converges strongly to $Q(f)$, where $Q : \Pi_C \to$* $\text{Fix}(T)$ *is defined by*

$$Q(f) := \lim_{t\to 0} x_t,$$

and x_t solves the fixed point equation $x_t = tf(x_t) + (1 - t)Tx_t$ for each $t \in (0, 1)$.

On the other hand, Kim and Xu [19] proposed the following simpler modification of the Mann iteration method:

Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a nonexpansive mapping such that $\text{Fix}(T) \ne \emptyset$. For an arbitrary $x_0 \in C$, define $\{x_n\}$ in the following way:

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ x_{n+1} = \beta_n u + (1 - \beta_n)y_n, \quad \forall n \ge 0, \end{cases} \tag{1.4}$$

where $u \in C$ is an arbitrary (but fixed) element in C, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$.

We remark that the modified Mann iteration scheme (1.4) is a convex combination of a fixed point in C and the Mann iteration method (1.2). There is no additional projection involved in iteration scheme (1.4). They proved a strong convergence theorem for iteration scheme (1.4).

Theorem KX (*[19, Theorem 1]*). *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X and let $T : C \to C$ be a nonexpansive mapping such that* $\text{Fix}(T) \ne \emptyset$. *Given a point $u \in C$ and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in* $(0, 1)$, *the following conditions are satisfied:*

(i) $\beta_n \to 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(ii) $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}_{n=0}^{\infty}$ defined by (1.4) *converges strongly to a fixed point of T.*