Nonlinear Analysis 73 (2010) 2117–2131

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Superlinear elliptic equations with singular coefficients on the boundary

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ARTICLE INFO

Article history:
Received 25 May 2009
Accepted 17 May 2010

MSC:
35J20
35J25
35J60

Keywords:
Superlinear elliptic equation
Singular coefficient
Positive solution
Infinitely many solutions
Variational method

ABSTRACT

In this paper, a superlinear elliptic equation whose coefficient diverges on the boundary is studied in any bounded domain Ω under the zero Dirichlet boundary condition. Although the equation has a singularity on the boundary, a solution is smooth on the closure of the domain. Indeed, it is proved that the problem has a positive solution and infinitely many solutions without positivity, which belong to $C^1(\overline{\Omega})$ or $C^2(\overline{\Omega})$. Moreover, it is proved that a positive solution has a higher order regularity up to $C^\infty(\overline{\Omega})$.

1. Introduction

We consider the superlinear elliptic equation with a singular coefficient,

$$\begin{cases} -\Delta u = h(x)|u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 2$, the nonlinear term is superlinear and subcritical, i.e., $1 < p < \infty$ if $N = 2$, $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $h(x)$ is a nonnegative measurable function in Ω. We study problem (1.1) when $h(x)$ may or may not have a singularity on the boundary. However we prove the existence of regular solutions on the whole $\overline{\Omega}$ which belong to $C^1(\overline{\Omega})$ or $C^2(\overline{\Omega})$. Indeed, although $h(x)$ may diverge to infinity as x tends to the boundary, a solution $u(x)$, if it exists, converges to zero because of the boundary condition. Then the right-hand side $h(x)|u|^{p-1}u$ of (1.1) may belong to $L^q(\Omega)$ with a certain q. Hence u lies in $W^{2,q}(\Omega)$ by the elliptic regularity theorem. If $q > N$, then u belongs to $C^1(\overline{\Omega})$ because of the Sobolev imbedding. Thus we can expect the existence of a regular solution on $\overline{\Omega}$.

We summarize the known results. Senba et al. [1] treated the case where Ω is a unit ball B and $h(x)$ has a power singularity, i.e., $h(x) = (1-|x|)^{-a}$ with $a > 0$. Then they proved that (1.1) has a positive radial solution in $C^2(B) \cap C^1(\overline{B})$ if $0 < a < 2$. Hashimoto and Ôtani [2,3] improved the result. Replacing the right-hand side of (1.1) by a general nonlinear term $f(|x|, u)$ having a singularity on $|x| = 1$, Lü and Bai [4], Usami [5] and Baxley [6] proved the existence of a positive radial solution in $C^2(B) \cap C(\overline{B})$. Another type of singularity, $f(x, u) = g(x)u^{-a}$ with $a > 0$ and $g \in C(\overline{\Omega})$ in a bounded domain Ω, is studied by Crandall et al. [7] and many authors [8–10, and the references therein]. In this case, the nonlinear term is singular at $u = 0$. Hence the zero Dirichlet condition yields a singularity on the boundary. On the other hand, our nonlinearity $h(x)u^p$ with $p > 1$ has a singularity which is caused by the behavior of $h(x)$ as x tends to the boundary. It seems to the author

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0362-546X/$ – see front matter
doi:10.1016/j.na.2010.05.039

that little is known about (1.1). As stated above, the papers [1–6] treat the ball and prove the existence of a radial solution. We emphasize that in this paper we deals with any bounded domain and study a general singular coefficient $h(x)$. Then we prove the existence of a positive solution and infinitely many solutions without positivity, which belong to $C^1(\overline{\Omega})$ or $C^2(\overline{\Omega})$ corresponding to the condition on $h(x)$. Moreover, we prove that a positive solution has a higher order regularity up to $C^\infty(\overline{\Omega})$. We have recently proved the same result as above in [11] for the sublinear case, $0 < p < 1$. However, the superlinear problem is more difficult than the sublinear one. Indeed, we use a variational method to get the result, but it is not easy to prove that the Lagrangian functional satisfies the Palais–Smale condition and the $H_0^1(\Omega)$-boundedness of solutions implies the $W^{2,q}(\Omega)$-boundedness. To prove them, we introduce new inequalities of the Hardy–Sobolev type, which include the singular integrals with the weight function diverging on the boundary.

We organize this paper into six sections. In Section 2, we state main results. In Section 3, we prove several integral inequalities of the Hardy–Sobolev type. In Section 4, we get a positive solution and infinitely many solutions without positivity in $H_0^1(\Omega)$. In Section 5, we prove the existence of solutions belonging to $W^{2,q}(\Omega)$ with $q > N$. Therefore these solutions have the $C^1(\overline{\Omega})$-regularity because of the Sobolev imbedding. In Section 6, we prove that any solution belongs to $C^2(\overline{\Omega})$ and a positive solution has a higher order regularity up to $C^\infty(\overline{\Omega})$ under suitable assumptions on $h(x)$.

2. Main results

First, we introduce the assumptions below.

(A1) Let $1 < p < \infty$ if $N = 2$ and $1 < p < (N + 2)/(N - 2)$ if $N \geq 3$.
(A2) Let $h(x)$ be a measurable function such that $h \geq 0$ and $h \not\equiv 0$ in Ω.
(A3) Suppose that there exist constants a and q such that $h(x)\rho(x)^a \in L^q(\Omega)$ and

$$0 \leq a, \quad 1 < q < \infty, \qquad 2a + \frac{2N}{q} < N + 2 - (N - 2)p, \tag{2.1}$$

where $\rho(x)$ is defined by

$$\rho(x) \equiv \mathrm{dist}(x, \partial\Omega) = \inf\{|x - y| : y \in \partial\Omega\}. \tag{2.2}$$

The assumption $h\rho^a \in L^q$ in (A3) allows $h(x)$ to have singularity on the boundary because $\rho(x)$ vanishes on $\partial\Omega$. We denote by $W^{m,q}(\Omega)$ the Sobolev space which consists of all $u \in L^q(\Omega)$ such that all the distributional derivatives up to order m lie in $L^q(\Omega)$. Let $W_0^{m,q}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{m,q}(\Omega)$. Put $H_0^1(\Omega) \equiv W_0^{1,2}(\Omega)$. Let $C^\theta(\overline{\Omega})$ denote the set of the Hölder continuous functions on $\overline{\Omega}$ with exponent θ. We define $C^{m,\theta}(\overline{\Omega})$ by the set of m times continuously differentiable functions whose mth order derivatives belong to $C^\theta(\overline{\Omega})$. We call u a $W^{2,q}(\Omega)$-solution if $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ and it satisfies (1.1) in the distribution sense.

Theorem 2.1. *Suppose that* (A1)–(A3) *hold. Then* (i) *and* (ii) *below hold.*

(i) *(1.1) has a nonnegative nontrivial $H_0^1(\Omega)$-solution.*
(ii) *(1.1) has a sequence $\{u_n\}$ of $H_0^1(\Omega)$-solutions whose $H_0^1(\Omega)$-norm diverges to infinity as $n \to \infty$.*

In the above theorem, the $H_0^1(\Omega)$-solution makes sense. Indeed, (A3) means that $h \in L^q_{loc}(\Omega)$. When $u \in H_0^1(\Omega)$, $|u|^{p-1}u$ belongs to $L^r(\Omega)$ with $r = 2N/((N - 2)p)$ for $N \geq 3$ and any number $r > 1$ for $N = 2$. Thus $h|u|^{p-1}u \in L^s_{loc}(\Omega)$ with $1/s = 1/q + 1/r$. Note that $s > 1$ by (2.1) if $N \geq 3$ and by taking $r > 1$ large enough if $N = 2$. Then the relation below makes sense,

$$-\int_\Omega u\Delta\phi \mathrm{d}x = \int_\Omega h|u|^{p-1}u\phi \mathrm{d}x,$$

for any $\phi \in C_0^\infty(\Omega)$. Thus a $H_0^1(\Omega)$-solution is well defined.

Theorem 2.2. *Suppose that* (A1)–(A3) *hold with $q > N$ and $0 \leq a \leq p$. Then all the conclusions in Theorem 2.1 are still valid after replacing $H_0^1(\Omega)$ by $W^{2,q}(\Omega)$. Moreover, a nonnegative solution in* (i) *of Theorem 2.1 becomes a positive solution.*

Remark 2.3. Because of the Sobolev imbedding, a $W^{2,q}(\Omega)$-solution with $q > N$ belongs to $C^1(\overline{\Omega})$. Hence all the solutions in Theorem 2.2 have $C^1(\overline{\Omega})$-regularity.

Theorem 2.4. *Let $0 < \theta < 1$ and $h\rho^p \in C^\theta(\overline{\Omega})$. Then any $W^{2,q}(\Omega)$-solution with $q > N$ belongs to $C^{2,\theta}(\overline{\Omega})$.*

Remark 2.5. (i) Let $1 < p < (N + 2)/N$. Then the assumption $h\rho(x)^p \in C^\theta(\overline{\Omega})$ implies (A3). Indeed, this assumption means clearly $h\rho^p \in L^q(\Omega)$ for any $q \geq 1$. Moreover, the condition $p < (N + 2)/N$ implies (2.1) by putting $a = p$ and by choosing q sufficiently large. Thus when $1 < p < (N + 2)/N$, the assumption $h\rho(x)^p \in C^\theta(\overline{\Omega})$ with (A1) and (A2) guarantees the existence of a $W^{2,q}(\Omega)$-solution, and therefore it belongs to $C^{2,\theta}(\overline{\Omega})$.
(ii) When $(N + 2)/N \leq p < (N + 2)/(N - 2)$, we assume $h\rho(x)^p \in C^\theta(\overline{\Omega})$ and (A1)–(A3) with $q > N$ and $a \leq p$. Then we have a positive $C^{2,\theta}(\overline{\Omega})$-solution and a sequence of $C^{2,\theta}(\overline{\Omega})$-solutions without positivity.