Nonlinear Analysis 73 (2010) 2148–2157

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Necessary optimality conditions for weak sharp minima in set-valued optimization

M. Durea [a,*], R. Strugariu [b]

[a] Faculty of Mathematics, "Al. I. Cuza" University, Bd. Carol I, nr. 11, 700506 – Iaşi, Romania

[b] Department of Mathematics, "Gh. Asachi" Technical University, Bd. Carol I, nr. 11, 700506 – Iaşi, Romania

ARTICLE INFO

Article history:
Received 8 October 2009
Accepted 14 May 2010

MSC:
90C29
90C26
49J52

Keywords:
Set-valued optimization
Sharp efficiency
Mordukhovich generalized differentiation
Fréchet normal cone

ABSTRACT

The aim of the present paper is to get necessary optimality conditions for a general kind of sharp efficiency for set-valued mappings in infinite dimensional framework. The efficiency is taken with respect to a closed convex cone and as the basis of our conditions we use the Mordukhovich generalized differentiation. We have divided our work into two main parts concerning, on the one hand, the case of a solid ordering cone and, on the other hand, the general case without additional assumptions on the cone. In both situations, we derive some scalarization procedures in order to get the main results in terms of the Mordukhovich coderivative, but in the general case we also carryout a reduction of the sharp efficiency to the classical Pareto efficiency which, in addition with a new calculus rule for Fréchet coderivative of a difference between two maps, allows us to obtain some results in Fréchet form.

1. Introduction and preliminaries

The sharp efficiency in scalar and vector optimizations has received a great amount of attention in the last decade. Among the first approaches in the literature is the study of Hestenes [1] for sharp efficiency for smooth scalar functions in finite dimensional setting. Then a new impetus of this direction of research has been given by the works of Ward [2], Ward and Studniarski [3]. A full discussion for convex real-valued functions is conducted in [4]. The first extensions to the situation of functions taking values in a normed vector space partially ordered by a closed convex cone was the object of the papers [5,6] and among the most recent effort in this study, including the case of set-valued maps, we can mention as well the papers [7–11].

Our aim in the present paper is to get necessary optimality conditions for a fairly general kind of sharp efficiency for set-valued mappings in infinite dimensional framework. The efficiency is taken with respect to a proper closed convex (not necessarily pointed) cone and as the basis of our condition we have chosen the Mordukhovich generalized differentiation objects because of the robustness of this theory and its rich calculus. The Fréchet constructions of normal cone, subdifferential and coderivative are also used as complementary tools. So, there are many differences between our results and the results in the recent papers we quoted above simply from these initial data and purposes and other more specific comparisons are given throughout the paper.

As usual in vector optimization, there is an important gap at both levels of the methods and the results between the case where the ordering cone is solid (i.e. with nonempty topological interior) and the general situation where we have no additional assumptions on the cone. For this reason, we have divided our work into two main parts concerning these two

* Corresponding author.
E-mail addresses: durea@uaic.ro (M. Durea), rstrugariu@tuiasi.ro (R. Strugariu).

0362-546X/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2010.05.041

different cases. In both situations, some scalarization procedures yield the main results, but in the solid case one can work directly with the initial set-valued map while in the other case one uses only the associated epigraphical set-valued map. Moreover, in some sense, the results in the solid case are sharper.

The paper is organized as follows. The second section contains the basic ingredients of our study. We describe here the setting of the paper, the definition of efficiency we consider, a comparison of it with other similar concepts in literature and the main constructions of generalized differentiation for sets, functions and set-valued maps along their main calculus properties. The third section contains the main results in the case where the ordering cone is solid. First, we characterize the efficiency concept be means of Gerstewitz's scalarizing functional and, together with several calculus rules for scalar functions, this allows us to get optimality conditions in terms of normal coderivatives of the reference set-valued map and its epigraphical set-valued map. The last section concentrates on the general case where the ordering cone is not necessarily solid. Here we observe that the very definition of sharp efficiency can be easily written as a natural minimality condition for a scalar function involving the epigraphical set-valued map associated with the underlying set-valued map. In this sense, we derive in a similar manner as in the preceding section, some optimality conditions which, however, differ from those in solid case in several aspects. Finally, as an attempt to bring closer the forms of sets of Lagrange multipliers in the two cases we derive some results in terms of Fréchet coderivative. To this aim, we develop a completely new technique based on two facts: the reduction of sharp efficiency to Pareto efficiency and an enhanced calculus rule for the Fréchet coderivative of the difference between a set-valued map and a single-valued map.

2. Basic concepts and results

Throughout the paper X, Y, Z are Banach spaces over the real field $\mathbb{R}$. Additional properties for these spaces will be stated explicitly, if needed. We denote by B_X, D_X and S_X the open unit ball, the closed unit ball and the unit sphere of X, respectively. $B(x,\varepsilon)$ and $D(x,\varepsilon)$ are the open and closed balls with center $x \in X$ and radius $\varepsilon > 0$ on the underlying space and X^* is the topological dual space of X. By w and w^* we mean the weak topology on X and the weak star topology on X^*, respectively. On a product space such $X \times Y$ we consider the sum norm. As usual, cl, int, bd denote the topological closure, the topological interior and the topological border of sets, respectively. We consider a proper closed convex cone $K \subset Y$ which introduces a preorder relation on Y by the equivalence $y_1 \leq_K y_2$ iff $y_2 - y_1 \in K$. We do not suppose that the cone is pointed (i.e. $K \cap -K = \{0\}$). We set $K^* := \{y^* \in Y^* \mid \langle y^*, y\rangle \geq 0, \forall y \in K\}$ for the dual cone of K.

Recall that with respect to the ordering cone K one can define the Pareto efficiency as follows. Let $A \subset Y$ be a nonempty subset of Y. A point $\overline{y} \in A$ is said to be a Pareto minimum point of A with respect to K if $(A - \overline{y}) \cap -K \subset K$. Note that if $\operatorname{int} K \neq \emptyset$, the point $\overline{y} \in A$ is said to be a weak Pareto minimum point of A with respect to K if $(A - \overline{y}) \cap - \operatorname{int} K = \emptyset$. Also, remark that if the cone K is pointed, the definition of Pareto minima reduces to the usual one (i.e. $(A - \overline{y}) \cap -K = \{0\}$). Concerning the weak Pareto minima, the relation $(A - \overline{y}) \cap - \operatorname{int} K = \emptyset$ is equivalent, because the cone K is proper, to $(A - \overline{y}) \cap - \operatorname{int} K \subset K$, whence in this case the expected definition corresponds to the classical one.

In the sequel we consider some set-valued maps $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$. As usual, the domain and the graph of F are $\operatorname{Dom} F = \{x \in X \mid F(x) \neq \emptyset\}$ and $\operatorname{Gr} F = \{(x, y) \mid y \in F(x)\}$, respectively. If $A \subset X$, $F(A) := \bigcup_{x\in A} F(x)$ and the inverse set-valued map of F is $F^{-1} : Y \rightrightarrows X$ given by $(y, x) \in \operatorname{Gr} F^{-1}$ iff $(x, y) \in \operatorname{Gr} F$.

In these notations, we consider a subset $S \subset X$ and the following optimization problem:

$$(\mathrm{P}_S) \quad \text{minimize } F(x), \quad \text{subject to } x \in S.$$

If $S = X$, we denote the (unconstrained) associated problem by (P).

The classical types of solutions of these problems with respect to the preorder given by K are defined as follows: a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is called a local Pareto minimum point for (P_S) if there exists a neighborhood U of $\overline{x}$ s.t. $\overline{y}$ is a Pareto minimum for $F(U \cap S)$. If $U = X$ then one says that $(\overline{x}, \overline{y})$ is a Pareto minimum point for (P_S). If $\operatorname{int} K \neq \emptyset$ then one can define in a similar manner the concept of weak Pareto minimum.

The main objective of this paper is to study another kind of efficiency in set-valued optimization, as defined below. Recall that if $y \in Y$ and $M \subset Y$ the oriented distance function is defined (cf. [12]) as $\Delta(y, M) := d(y, M) - d(y, Y \setminus M)$ where, as usual, $d(y, M) := \inf_{u\in M} \|y - u\|$.

Definition 2.1. Let $\varepsilon > 0$ and $\psi : (-\varepsilon, +\infty) \to \mathbb{R}$ be a nondecreasing function on $[0, +\infty)$ with the property that $\psi(t) = 0$ if and only if $t = 0$. One says that a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (S \times Y)$ is a weak ψ-sharp local Pareto minimizer for (P_S), if there exists $\alpha > 0$ and a neighborhood U of $\overline{x}$ such that for every $x \in U \cap S$, $y \in F(x)$ one has

$$\Delta(y - \overline{y}, -K) \geq \alpha\psi(d(x, W)), \tag{1}$$

where $W := \{x \in S \mid \overline{y} \in F(x)\} = S \cap F^{-1}(\overline{y})$.

If $W = \{\overline{x}\}$ and one takes $\psi(t) = t$, then relation (1) becomes: for every $x \in U \cap S$, $y \in F(x)$ one has

$$\Delta(y - \overline{y}, -K) \geq \alpha \|x - \overline{x}\|,$$

and in this case one says that $(\overline{x}, \overline{y})$ is a local sharp minimizer for (P_S).