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Nonexistence, existence and multiplicity of positive solutions to the $p(x)$-Laplacian nonlinear Neumann boundary value problem[☆]

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ABSTRACT

We study the nonexistence, existence and multiplicity of positive solutions for the nonlinear Neumann boundary value problem involving the $p(x)$-Laplacian of the form

$$\begin{cases} -\Delta_{p(x)}u + \lambda|u|^{p(x)-2}u = f(x,u) & \text{in } \Omega \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \eta} = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbf{R}^N$, $p \in C^1(\overline{\Omega})$ and $p(x) > 1$ for $x \in \overline{\Omega}$. Using the sub–supersolution method and the variational principles, under appropriate assumptions on f and g, we prove that there exists $\lambda_* > 0$ such that the problem has at least two positive solutions if $\lambda > \lambda_*$, has at least one positive solution if $\lambda = \lambda_*$ and has no positive solution if $\lambda < \lambda_*$.

1. Introduction

The purpose of this paper is to study the nonexistence, existence and multiplicity of positive solutions for the following nonlinear Neumann boundary value problem involving the $p(x)$-Laplacian

$$(N_\lambda) \quad \begin{cases} -\Delta_{p(x)}u + \lambda|u|^{p(x)-2}u = f(x,u) & \text{in } \Omega \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \eta} = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbf{R}^N$ with $C^{1,\beta}$ smooth boundary, η is the unit outward normal to $\partial\Omega$, $\lambda \in \mathbf{R}$, $p \in C^1(\overline{\Omega})$ with $p^- := \inf_{x\in\overline{\Omega}} p(x) > 1$. And throughout this paper the following assumption holds:

Assumption (FG).

(1) $f \in C(\overline{\Omega} \times \mathbf{R})$ and

$$f(x,t) \geq 0, \quad \forall x \in \overline{\Omega}, \forall t \geq 0.$$

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(2) $g \in C(\partial\Omega \times \mathbf{R})$ and on $\partial\Omega$

$$\forall t \geq 0, \quad g(x,t) \geq 0 \quad \text{and} \quad g(x,0) \not\equiv 0.$$

(3) for any $x_1, x_2 \in \partial\Omega$, $t_1, t_2 \in \mathbf{R}$,

$$|g(x_1,t_1) - g(x_2,t_2)| \leq \Lambda(\max\{|t_1|,|t_2|\}) \left(|x_1 - x_2|^{\beta_1} + |t_1 - t_2|^{\beta_2}\right)$$

where $\Lambda : [0,+\infty) \to (0,+\infty)$ is a nondecreasing continuous function and $\beta_1, \beta_2 \in (0,1)$.

The operator $-\Delta_{p(x)}u := -\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, which becomes p-Laplacian when $p(x) \equiv p$ (a constant). The study of various mathematical problems with variable exponent has been received considerable attention in recent years. These problems are interesting in applications (see e.g. [1,2]) and raise many difficult mathematical problems. We refer to [3,4] for the overview and references of this subject, and to [5–16] for the study of the $p(x)$-Laplacian equations and the corresponding variational problems.

Many authors have studied the inhomogeneous Neumann boundary value problems involving the p-Laplacian, see e.g. [17–22] and the references therein.

In [18] the authors have studied the inhomogeneous Neumann boundary value problem:

$$\begin{cases} -\Delta u + \lambda u = u^q & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \eta} = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.1$_\lambda$}$$

where $1 < q \leq 2^* - 1 = \frac{N+2}{N-2}$ with $N > 2$.

The authors of [17] have investigated the following problem:

$$\begin{cases} -\Delta_p u + \lambda u^{p-1} = u^q & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \eta} = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.2$_\lambda$}$$

where $p - 1 < q \leq p^* - 1 =: \frac{Np}{N-p} - 1$ with $1 < p < N$.

In the present paper the function $f(x,u)$ of the problem (N_λ) is of general type which includes $u^{q(x)}$ (with $u > 0$) as a special case. And in [17,18] only the case that $f(x,u) = u^q$ (with $u > 0$) is considered. In studying the existence of positive solutions of (N_λ) we do not restrict the growth condition of $f(x,u)$ and $g(x,u)$, but in studying the multiplicity of positive solutions of (N_λ), in this paper, we shall restrict ourselves to the case that $f(x,u)$ and $g(x,u)$ satisfy the subcritical growth conditions because the study of the $p(x)$-Laplacian equations in the critical growth case is very difficult and requires some special preliminaries which are not ready up to the present.

The special features of this class of problems, considered in the present paper, are that they involve the variable exponent and nonlinear boundary conditions. The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian. For example, it is inhomogeneous. In this paper we will generalize the main results of [17,18] to the $p(x)$-Laplacian case and also generalize the case of inhomogeneous Neumann boundary conditions to nonlinear Neumann boundary conditions case. The elliptic equation problems with nonlinear boundary conditions have attracted many authors' interest in recent years, and they are well known in the literature. For example, for the Laplacian with nonlinear boundary conditions see [23–26], for elliptic systems with nonlinear boundary conditions see [27,28], for the p-Laplacian with nonlinear boundary conditions of different type see [29–33].

The main method used in this paper is the sub–supersolution method for the Neumann problems involving the $p(x)$-Laplacian, which is similar to that given in [8] for the Dirichlet problems involving the $p(x)$-Laplacian.

Let

$$\Lambda = \{\lambda \in \mathbf{R} : \text{ there exists at least a positive solution of the problem}(N_\lambda)\},$$
$$\lambda_* = \inf \Lambda.$$

Firstly, applying the theory of variable exponent Sobolev spaces, established first by Kováčik and Rákosník [34], and some research results obtained recently for the $p(x)$-Laplacian equations, in particular, the results of [9] on the global $C^{1,\alpha}$ regularity of the weak solutions for the $p(x)$-Laplacian equations, we obtain that $(\lambda_*, +\infty) \subset \Lambda$ and there exists a minimal positive solution of (N_λ) if $\lambda > \lambda_*$ (see Theorem 4.1 in Section 4). We also prove that there exists a positive solution of (N_λ) which is a local minimizer of the energy functional associated with (N_λ) if $\lambda > \lambda_*$(see Theorem 4.2 in Section 4). A main difficulty for proving Theorem 4.2 in Section 4 is that a special strong comparison principle is required. It is well known that, when $p \neq 2$, the strong comparison principles for the p-Laplacian equations are very complicated (see e.g. [35–38]). In [39,40,8] the required strong comparison principles for the Dirichlet problems have be established, however, they cannot be applied to the Neumann problems. To prove this theorem, we establish a special strong comparison principle for the Neumann problem (N_λ) (see Lemma 3.3 in Section 3), which is also valid for the homogeneous and inhomogeneous Neumann boundary value problems.