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The blow-up phenomenon for degenerate parabolic equations with variable coefficients and nonlinear source

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ABSTRACT

The Cauchy problem for a degenerate parabolic equation with a source and variable coefficient of the form

$$\frac{\partial u}{\partial t} = div(\rho(x)u^{m-1}|Du|^{\lambda-1}Du) + u^p$$

is studied. Global in time existence and nonexistence conditions are found for a solution to the Cauchy problem. Exact estimates of a solution are obtained in the case of global solvability. A sharp universal (i.e., independent of the initial function) estimate of a solution near the blow-up time is obtained.

1. Introduction

This paper is devoted to the study of the asymptotic in time behavior of the nonnegative solutions to the Cauchy problem

$$\frac{\partial u}{\partial t} = div(\rho(x)u^{m-1}|Du|^{\lambda-1}Du) + u^p, \tag{1}$$

$$(x,t) \in Q_T = \mathbb{R}^N \times (0,T), \quad T > 0, N \geq 1,$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n. \tag{2}$$

In what follows it is assumed that $\lambda > 0$ and $m+\lambda-2 > 0$, that is we are in the range of slow diffusion and the variable coefficient $\rho(x) > 0$ is either $\rho(x) = |x|^\alpha$ for $0 < \alpha < \lambda+1$ or $\rho(x) = (1+|x|)^\alpha$ for $\lambda+1 \leq \alpha < \infty$. Next we assume that $p > m+\lambda-1$. Moreover, $u_0(x)$ is a nonnegative measurable function belonging to $L_{1,loc}(R^N)$ and such that

$$\int_{R^N} u_0^{1+\frac{m+\lambda-1}{\lambda}} dx < \infty.$$

The problem (1)–(2), but without the blow-up term has been studied in [1] where among other results it was proven that if $\rho(x) = (1+|x|)^\alpha$, $\lambda+1 < \alpha$, then the long-time behavior has an universal structure, that is

$$\|u(x,t)\|_{\infty,\mathbb{R}^N} \leq C^* t^{-\frac{1}{m+\lambda-2}} \quad \forall t \in (0,\infty), \tag{3}$$

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where the constant C^* does not depend on u_0. This is a surprising fact, since, such a behavior is typical for the corresponding Cauchy–Dirichlet problem in bounded domains. Notice, that the estimate (3) for the problem

$$(1+|x|)^{-l}\frac{\partial v}{\partial t} = div(v^{m-1}|Dv|^{\lambda-1}Dv) + (1+|x|)^{-l}v^p, \tag{4}$$

$$(x,t) \in Q_T = \mathbb{R}^N \times (0,T), \quad T > 0, N \geq 1,$$

$$v(x,0) = v_0(x_0), \quad x \in \mathbb{R}^n, \tag{5}$$

was obtained in [2] when $l > \lambda + 1$ and $p > m + \lambda - 1$.

The main motivation of the present paper is to get (3) for a solution of (1)–(2) with $p > m + \lambda - 1$. Besides, we will examine the case $0 < \alpha < \lambda + 1$ to get the Fujita type result and to investigate the behavior near the blow-up time as well.

In [3] it has been studied an equation similar to (4), but in a more general setting though without blow-up term, proving the Holder continuity of generalized solutions. Namely in [3] the authors looked at the problem

$$\rho(x)\frac{\partial u}{\partial t} - \sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,t,u,Du) = a_0(x,t,u,Du), \tag{6}$$

where the Caratheodory functions $a_i(x,t,u,\zeta) : \Omega_T \times R \times R^N \to R,\ i = 0, 1, \ldots, n$ satisfy the structural conditions

$$\sum_{i=1}^{N} a_i(x,t,u,\zeta)\zeta_i \geq C_0|u|^{m-1}|\zeta|^{\lambda+1} - \varphi_0(x,t),$$

$$|a_i(x,t,u,\zeta)| \leq C_1|u|^{m-1}|\zeta|^{\lambda} + |u|^{\frac{m-1}{\lambda+1}}\varphi_1(x,t), \quad i = 1, \ldots, n,$$

$$|a_0(x,t,u,\zeta)| \leq C_2|u|^{m-1}|\zeta|^{\lambda+1} + \varphi_2(x,t),$$

and $\varphi_i, i = 0, 1, 2$ are measurable functions belonging to some L_q spaces (see for details [3]).

The interface blow-up phenomenon was investigated in [4] for inhomogeneous porous media equation (IPME) and in [5], [1] for (6). L_1–L_∞ estimates of solutions of the Cauchy problem for an anisotropic equation were studied in [6]. The asymptotic behavior of solutions and uniqueness results for IPME were studied recently in several papers [7–9,2,3,10–12]. For Cauchy problem for parabolic equation see also [13–17].

Next we introduce the notion of a generalized solution. Note that (1) can be written in the equivalent form

$$\frac{\partial v^\beta}{\partial t} = \beta^\lambda div(\rho(x)|Dv|^{\lambda-1}Dv) + v^\mu,$$

where $\beta = \frac{\lambda}{m+\lambda-1}$, $\mu = \frac{\lambda p}{m+\lambda-1}$, $u = v^\beta$.

Definition 1. We say that $u(x,t)$ is a generalized solution (or simply a solution) to problem (1)–(2) in $Q_T = \mathbb{R}^N \times (0,T)$ if

$$v \in L^{\mu+1}((0,T)\times\mathbb{R}^N), \qquad v^{\beta+1} \in C([0,T), L^1(\mathbb{R}^N)), \qquad \rho(x)|Dv|^{\lambda+1} \in L^1(Q_T),$$

$$\|v(x,t)^{\beta+1} - v_0(x)^{\beta+1}\|_{L^1(\mathbb{R}^N)} \to 0 \quad \text{as } t \to 0$$

and

$$\int_0^T\int_{\mathbb{R}^N}\left\{-v^\beta\varphi_t + \beta^\lambda\rho(x)|Dv|^{\lambda-1}DvD\varphi\right\}\mathrm{d}x\mathrm{d}t = \int_0^T\int_{\mathbb{R}^N} v^\mu\varphi\mathrm{d}x\mathrm{d}t,$$

for any testing function $\varphi(x,t) \in C_0^1(Q_T)$.

Existence of local in time solutions to (1)–(2) follows from results of [18,19].

The problem of uniqueness of solutions for (1)–(2) when

$$\alpha \geq N(m+\lambda-2) + \lambda + 1$$

is still open. In this case our results apply to the solution we construct in Theorem 2 that is as limit of corresponding family of solutions to Cauchy–Dirichlet problem in bounded strips. While if our weighted functions is $(1+|x|)^\alpha$, $\lambda + 1 \leq \alpha < N(m+\lambda-2) + \lambda + 1$ and the initial datum is compactly supported then as it follows from results in [20], the solution of (1)–(2) is compactly supported as well for all $t > 0$ and therefore may be considered as a solution of Cauchy–Dirichlet problem in a bounded cylinder. Thus the uniqueness in this case is done. Finally, if our weighted function has a form $|x|^\alpha$ with $0 \leq \alpha < \lambda + 1$, then uniqueness follows from [11].

Remark 1.1. To prove the basic results, we will use a local energy approach which is flexible enough and, therefore, Eq. (1) does not need to be self-similar. Furthermore, all the results are valid under more general assumptions on $\rho(x)$. In particular, we may consider $\rho(x)$ as a measurable function such that $\rho(x) \sim |x|^\alpha$, or more generally we may assume that $\rho(x)$ belongs to Muckenhoupt class with some additional assumptions on $\rho(x)$ (see for details [3]).