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Asymptotically self-similar solutions of the damped wave equation

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ABSTRACT

We consider the damped hyperbolic equation

$$\varepsilon u_{\tau\tau} + u_\tau = (a(\xi)u_\xi)_\xi - |u|^{p-1}u, \quad (\xi, \tau) \in \mathbb{R} \times \mathbb{R}_+, \tag{1}$$

where $\varepsilon > 0$, $a(\xi) \to 1$ as $|\xi| \to +\infty$ and $1 < p < 3$.
We prove in this article that the exact self-similar solutions of the semi-linear parabolic equation obtained by setting $\varepsilon = 0$ and $a(\xi) \equiv 1$ in (1) are also asymptotically stable self-similar solutions of the Eq. (1). The proof of our result relies on various energy estimates rewritten in the variables $\xi/\sqrt{\tau}$, $\ln \tau$.

1. Introduction

In this paper, we study the asymptotic stability of the solution of the damped hyperbolic equation given as follows:

$$\varepsilon u_{\tau\tau} + u_\tau = (a(\xi)u_\xi)_\xi - |u|^{p-1}u, \quad (\xi, \tau) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

where ε is a positive, (not necessarily small) parameter, and $p \in]1, 3[$. We assume that the diffusion coefficient $a(\xi)$ is positive and satisfies $\lim_{\xi \to \pm\infty} a(\xi) = 1$.

Equations of type (1.1) are commonly used as mathematical models in several fields dealing with spreading and interacting particles: especially in genetics and population dynamics; see for example [1,2]. In the case of an inhomogeneous medium, the diffusion coefficients in such equations may depend on the space variable ξ.

In the case $p > 3$, Gallay and Raugel [3] have studied the stability of small solutions of Eq. (1.1) under the hypothesis $\lim_{\xi \to \pm\infty} a(\xi) = a_\pm > 0$. In fact, they introduced scaling variables and used energy estimates to show that the asymptotic behavior of a small solution $u(\xi, \tau)$ of Eq. (1.1) is entirely determined, up to the second order, by a linear parabolic equation depending only on $a_\pm$.

In the case $\varepsilon = 0$ and $a(\xi) \equiv 1$, Eq. (1.1) reduces to the following semi-linear parabolic equation:

$$u_\tau = u_{\xi\xi} - |u|^{p-1}u. \tag{1.2}$$

The asymptotic behavior of Eq. (1.2) has been rather extensively studied. We here recall some known results. First, we remark that if $u(\xi, \tau)$ is a solution of Eq. (1.2), then for all $\lambda > 0$, $u_\lambda(\xi, \tau) = \lambda^{\frac{2}{p-1}} u(\lambda^2\tau, \lambda\xi)$ is also a solution. A solution $u \not\equiv 0$ is said to be self-similar, when $u_\lambda \equiv u$, for all $\lambda > 0$.

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In the N multidimensional case, with $1 < p < 1 + \frac{2}{N}$, we already know that if Eq. (1.2) has a self-similar solution with initial data u_0, then u_0 is homogenous of degree $\frac{2}{1-p}$. On the other hand, Cazenave, Dickstein, Escobedo and Weissler [4] have shown, among many other results, that there exists a unique self-similar solution $\widetilde{u}$ of Eq. (1.2) such that

$$\lim_{|\xi|\to+\infty} |\xi|^{\frac{2}{p-1}} \left(\widetilde{u}(\xi,1) - u_0(\xi)\right) = 0. \tag{1.3}$$

These self-similar solutions are related to the asymptotic behavior of u solution of Eq. (1.2) in the following sense: suppose that there exist ω, a homogenous function of degree 0, and $u_0 \in \mathcal{C}(\mathbb{R})$ an asymptotically homogenous function in space in the following sense:

$$\lim_{|\xi|\to+\infty} |\xi|^{\frac{2}{p-1}} u_0(\xi) - \omega\left(\frac{\xi}{|\xi|}\right) = 0. \tag{1.4}$$

Assume in addition that $u_0 \geq 0$, $u_0 \not\equiv 0$. Then, the solution u of Eq. (1.2) is asymptotically self-similar in the sense that

$$\lim_{\tau\to+\infty} \sup_{\xi\in\mathbb{R}} \left((\tau + |\xi|^2)^{\frac{1}{p-1}} \left(u(\xi,\tau) - \widetilde{u}(\xi,\tau)\right)\right) = 0. \tag{1.5}$$

Recently, in [5], the authors proved that if the initial data does not behave asymptotically like the profile, as in (1.4), in a certain precise way, then the solution is not asymptotically self-similar. In the same way, an instability result has been established in [6,7].

Returning now to the one-dimensional case, a self-similar solution of Eq. (1.2) has the form $u(\xi,\tau) = \tau^{-\frac{1}{p-1}} f\left(\frac{\xi}{\sqrt{\tau}}\right)$, where f satisfies an ordinary differential equation

$$f''(x) + \frac{x}{2} f'(x) + \frac{1}{p-1} f(x) - |f(x)|^{p-1} f(x) = 0, \quad \text{in } \mathbb{R}_+. \tag{1.6}$$

Here we restrict our study to positive self-similar solutions satisfying $f'(0) = 0$. As shown in [8–10], for $p \in]1,3[$ and $\gamma \in [\gamma_p, \gamma^*[$, where $0 < \gamma_p < \gamma* = (p-1)^{-\frac{1}{p-1}}$, there exists a smooth, even, positive solution f_γ of Eq. (1.6) such that $f_\gamma(0) = \gamma$. Moreover, the function

$$\begin{aligned} c : [\gamma_p, \gamma^*[&\longrightarrow [0,+\infty[\\ \gamma &\longmapsto \lim_{x\to+\infty} x^{\frac{2}{p-1}} f_\gamma(x), \end{aligned} \tag{1.7}$$

is a bijection. On the other hand, the decay at infinity of these solutions f_γ is as follows (see [8]):

$$\begin{aligned} &\text{for } \gamma > \gamma_p, \quad f_\gamma(x) = c(\gamma)|x|^{\frac{-2}{p-1}} + \mathcal{O}\left(|x|^{\frac{-2}{p-1}-2}\right), && \text{as } |x| \to +\infty, \\ &\text{for } \gamma = \gamma_p, \quad f_\gamma(x) = c(\gamma_p)|x|^{\frac{2}{p-1}-1} e^{-\frac{x^2}{4}} + \mathcal{O}\left(|x|^{\frac{2}{p-1}-3} e^{-\frac{x^2}{4}}\right), && \text{as } |x| \to +\infty, \end{aligned} \tag{1.8}$$

where $c(\gamma)$ is defined by (1.7) and $c(\gamma_p)$ is a positive constant. Moreover,

$$\text{for } \gamma > \gamma_p, \quad \frac{f'_\gamma(x)}{f_\gamma(x)} = \frac{-2}{p-1}|x|^{-1} + \mathcal{O}(|x|^{-3}), \quad \text{as } |x| \to +\infty. \tag{1.9}$$

The stability of these self-similar solutions has been widely studied (see for example [11,12,10,13,14]).

For instance, using the properties (1.8), Escobedo, Kavian and Matano [9] have shown that, for $\gamma \in [\gamma_p, \gamma^*[$, for initial data $u(\xi,1)$, at $\tau = 1$ near $f_\gamma(\xi)$ in the space $L^\infty\left(\mathbb{R}, \left(1 + |x|^{\frac{2}{p-1}}\right)\right)$ and $\lim_{|\xi|\to+\infty} |\xi|^{\frac{2}{p-1}} u(\xi,1) = \lim_{|\xi|\to+\infty} |\xi|^{\frac{2}{p-1}} f_\gamma(\xi)$, there exists a unique global classical solution $u(\xi,\tau)$ of (1.2) $u(\xi,\tau)$ in the space $L^\infty\left([0,+\infty[, L^\infty\left(\mathbb{R}, \left(1 + |x|^{\frac{2}{p-1}}\right)\right)\right)$, which moreover satisfies

$$\lim_{\tau\to+\infty} \left\| \tau^{\frac{1}{p-1}} u(\xi\sqrt{\tau},\tau) - f_\gamma(\xi) \right\|_{L^\infty} = 0. \tag{1.10}$$

In a more recent paper, Bricmont and Kupiainen [15] have shown, that if $\gamma \in [\gamma_p, \gamma^*[$ and if the initial data $u(\xi,1)$ are near $f_\gamma(\xi)$ in the weighted space $L^\infty(\mathbb{R}, (1 + |x|^q))$, with $q > \frac{2}{p-1}$, Eq. (1.2) has a unique global classical solution $u(\xi,\tau)$ in the space $L^\infty([0,+\infty[, L^\infty(\mathbb{R}, (1 + |x|^q)))$. Moreover, there exist $\mu > 0$ and $C > 0$ such that, for all $\tau \geq 1$,

$$\left\| \left[\tau^{\frac{1}{p-1}} u(\xi\sqrt{\tau},\tau) - f_\gamma(\xi)\right] (1 + |\xi|^q) \right\|_{L^\infty} \leq C\tau^{-\mu} \|[u(\xi,1) - f_\gamma(\xi)](1 + |\xi|^q)\|_{L^\infty}. \tag{1.11}$$

In the whole paper, we fix a positive number γ in $]\gamma_p, \gamma^*[$, and we denote by g_γ the self-similar positive solution $g_\gamma(\xi,\tau) = \tau^{-\frac{1}{p-1}} f_\gamma\left(\xi\tau^{-\frac{1}{2}}\right)$ of Eq. (1.2).