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Construction of coupled Harry Dym hierarchy and its solutions from Stäckel systems

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ABSTRACT

In this paper we show how to construct the coupled (multicomponent) Harry Dym (cHD) hierarchy from classical Stäckel separable systems. Both nonlocal and purely differential parts of hierarchies are obtained. We also construct various classes of solutions of cHD hierarchy from solutions of corresponding Stäckel systems.

1. Introduction

Various relations between finite- and infinite-dimensional nonlinear integrable systems have been investigated since the middle of 70s in a long sequence of papers starting from paper [1], through papers [2–5] (see for example [6] for more detailed bibliography) and many others. In all these efforts, however, the main idea was to pass from infinite- to finite-dimensional integrable systems. This paper is a third paper in our series of papers showing that an opposite way is also possible: that of passing from ordinary differential equations integrable in the sense of Arnold–Liouville to infinite-dimensional integrable systems (soliton hierarchies). In paper [7] we demonstrated a way of generating commuting evolutionary flows from corresponding family of Stäckel systems (that is classical finite-dimensional Hamiltonian systems quadratic in momenta and separable in the sense of Hamilton–Jacobi theory). We presented our idea in the setting of coupled (multicomponent) KdV hierarchies (for definition and properties of these hierarchies, see for example [8]). In paper [9] we systematized and developed this idea by showing how solutions of these Stäckel systems can be used for generating various classes of solutions of cKdV hierarchies. Although both papers have been written for the case of cKdV, similar constructions are possible for other hierarchies as well. In this paper we demonstrate a way of generating the coupled (i.e. multicomponent) Harry Dym (cHD) hierarchy (see [10,11]) and various classes of its solutions from a class of Stäckel systems of Benenti type. Our method leads both to the nonlocal cHD hierarchy as well as to purely differential cHD hierarchy, that is to a multicomponent generalization of HD hierarchy discussed in [12] (see also [13]). The nonlocal part of cHD hierarchy has not been discussed in [10] at all. We also clarify and simplify some of the results given in [7,9].

The paper is organized as follows. In Section 2 we briefly remind some basic fact about Stäckel separable systems and discuss how they are related to corresponding Killing systems (dispersionless nonlinear PDE's of evolutionary type defined by Killing tensors of Stäckel systems). Sections 3 and 4 are devoted to the description of nonlocal multicomponent Harry Dym hierarchy and its various solutions, respectively. Sections 5 and 6 are devoted to local (purely differential) cHD hierarchy.

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2. Stäckel systems and their dispersionless counterpart

Stäckel separable systems can be most conveniently obtained from an appropriate class of separation relations. Generally speaking, n equations of the form

$$\varphi_i(\lambda_i, \mu_i, a_1, \ldots, a_n) = 0, \quad i = 1, \ldots, n, \ a_i \in \mathbf{R} \tag{1}$$

(each involving only one pair λ_i, μ_i of canonical coordinates on a $2n$-dimensional Poisson manifold $\mathcal{M}$) are called separation relations [14] provided that $\det\left(\frac{\partial\varphi_i}{\partial a_j}\right) \neq 0$. We can then locally resolve Eq. (1) with respect to a_i obtaining

$$a_i = H_i(\lambda, \mu), \quad i = 1, \ldots, n \tag{2}$$

with some new functions (Hamiltonians) $H_i(\lambda, \mu)$ that in turn generate n canonical Hamiltonian systems on $\mathcal{M}$:

$$\lambda_{t_i} = \frac{\partial H_i}{\partial \mu}, \qquad \mu_{t_i} = -\frac{\partial H_i}{\partial \lambda}, \quad i = 1, \ldots, n. \tag{3}$$

All the flows (3) mutually commute since the Hamiltonians H_i Poisson commute. Moreover, Hamilton–Jacobi equations for all the Hamiltonians H_i are separable in the (λ, μ)-variables since they are algebraically equivalent to the separation relations (1).

In this article we consider a special but important class of separation relations, namely

$$\sum_{j=1}^{n} a_j \lambda_i^{n-j} = \lambda_i^m \mu_i^2 + \frac{\varepsilon}{4}\lambda_i^k, \quad i = 1, \ldots, n \tag{4}$$

with arbitrary fixed $m, k \in \mathbf{Z}$,$\varepsilon = \pm 1$ (the constant $\frac{1}{4}$ is not essential for the construction and is only introduced for a smoother identification our systems with the hierarchy in [10]). The relations (4) are linear in the coefficients a_i so that they can be (globally) solved by Cramer formulas, which yields

$$a_i = \mu^T K_i G^{(m)} \mu + \frac{\varepsilon}{4} V_i^{(k)} \equiv H_i^{n,m,k}, \quad i = 1, \ldots, n, m, k \in \mathbf{Z} \tag{5}$$

where we denote $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ and $\mu = (\mu_1, \ldots, \mu_n)^T$. Functions H_i defined as the right-hand sides of (5) depend on m and k and can be interpreted as n quadratic in momenta μ Hamiltonians on the phase space $\mathcal{M} = T^*\mathcal{Q}$ cotangent to a Riemannian manifold $\mathcal{Q}$ parametrized by $(\lambda_1, \ldots, \lambda_n)$ and equipped with the contravariant metric tensor $G^{(m)}$ (depending on $m \in \mathbf{Z}$) given by:

$$G^{(m)} = \operatorname{diag}\left(\frac{\lambda_1^m}{\Delta_1}, \ldots, \frac{\lambda_n^m}{\Delta_n}\right) \quad \text{with } \Delta_i = \prod_{j\neq i}(\lambda_i - \lambda_j). \tag{6}$$

It can be shown that $G^{(m)}$ is of zero curvature for $m = 0, \ldots, n$ and that $G^{(n+1)}$ is of non-zero constant curvature, while all other choices of m lead to spaces of non-constant curvature. The Hamiltonians $H_i^{n,m,k}$ are known in literature as Stäckel Hamiltonians and the corresponding commuting Hamiltonian flows (3) are then called Stäckel systems, or more precisely, Stäckel systems of Benenti type. They are obviously separable in the sense of Hamilton–Jacobi theory since they by the very definition satisfy Stäckel relations (4). The objects K_i in (5) are Killing tensors for any metric $G^{(m)}$ and are given by

$$K_i = -\operatorname{diag}\left(\frac{\partial q_i}{\partial \lambda_1}, \ldots, \frac{\partial q_i}{\partial \lambda_n}\right) \quad i = 1, \ldots, n,$$

where $q_i = q_i(\lambda)$ are Viète polynomials (signed symmetric polynomials) in λ:

$$q_i(\lambda) = (-1)^i \sum_{1\leq s_1 < s_2 < \cdots < s_i \leq n} \lambda_{s_1} \ldots \lambda_{s_i}, \quad i = 1, \ldots, n \tag{7}$$

that can also be considered as new coordinates on the Riemannian manifold $\mathcal{Q}$ (we will then refer to them as Viète coordinates). Notice that K_i do not depend on neither m nor k. Finally, the potentials $V_i^{(k)}$ can be constructed recursively [15] by

$$V_i^{(k+1)} = V_{i+1}^{(k)} - q_i V_1^{(k)}, \quad k \in \mathbf{Z}, \text{ with } V_i^{(0)} = \delta_{in}, \tag{8}$$

where we put $V_i^{(k)} = 0$ for $i < 0$ or $i > n$. The first potentials are trivial: $V_i^{(k)} = \delta_{i,n-k}$ for $k = 0, 1, \ldots, n-1$. The first nontrivial potentials are $V_i^{(n)} = -q_i$. For $k > n$ the potentials $V_i^{(k)}$ become complicated polynomial functions of q. The recursion (8) can also be reversed

$$V_r^{(k)} = V_{r-1}^{(k+1)} - \frac{q_{r-1}}{q_n} V_n^{(k+1)}, \quad k \in \mathbf{Z}, r = 1, \ldots, n, \tag{9}$$