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Endpoints of multi-valued generalized weak contraction mappings

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ABSTRACT

Let (X, d) be a complete metric space, and let $T : X \to P_{cl,bd}(X)$ be a multi-valued generalized weak contraction mapping. Then T has a unique endpoint if and only if T has the approximate endpoint property. Our results extend previous results given by Ćirić (1971) [15], Nadler (1969) [11], Daffer and Kaneko (1995) [9] and Amini-Harandi (2010) [8].

1. Introduction and preliminaries

Let (X, d) be a metric space and $P(X)$ denote the class of all subsets of X. Define

$$P_f(X) = \left\{A \subseteq X : A \neq \emptyset \text{ has property } f\right\}.$$

Thus $P_{bd}(X)$, $P_{cl}(X)$, $P_{cp}(X)$ and $P_{cl,bd}(X)$ denote the classes of bounded, closed, compact, and closed bounded subsets of X, respectively. Also $T : X \to P_f(X)$ is called a multi-valued mapping on X. A point x is called a fixed point of T if $x \in Tx$. Define $\mathrm{Fix}(T) = \{x \in X : x \in Tx\}$. An element $x \in X$ is said to be an endpoint of a multi-valued mapping T, if $Tx = \{x\}$. We denote the set of all endpoints of T by $\mathrm{End}(T)$. The investigation of endpoints of multi-valued mappings has received great attention in recent years (see [1–8]). A mapping $T : X \to X$ is said to be a weak contraction if there exists $0 \le \alpha < 1$ such that

$$d(Tx, Ty) \le \alpha N(x, y), \tag{1.1}$$

for all $x, y \in X$, where

$$N(x, y) := \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}. \tag{1.2}$$

A multi-valued mapping $T : X \to P_{cl,bd}(X)$ is said to be a weak contraction if there exists $0 \le \alpha < 1$ such that

$$H(Tx, Ty) \le \alpha N(x, y), \tag{1.3}$$

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for all $x, y \in X$, where H denotes the Hausdorff metric on $P_{\mathrm{cl,bd}}(X)$ induced by d, that is,

$$H(A, B) := \max\left\{\sup_{x\in B} d(x, A), \sup_{x\in A} d(x, B)\right\}, \tag{1.4}$$

for all $A, B \in P_{\mathrm{cl,bd}}(X)$, and where

$$N(x, y) := \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}, \tag{1.5}$$

where $d(a, A) = \mathrm{dist}(a, A)$ for all $a \in X$ and all $A \in P_{\mathrm{cl,bd}}(X)$. The concept of weak contraction mappings was defined by Daffer and Kaneko [9] in 1995.

Many authors have studied fixed points for multi-valued mappings. Among many other studies, see, for example [10–13] and the references therein.

In the following theorem, Nadler [11] extended the Banach contraction principle to multi-valued mappings.

Theorem 1.1. *Let (X, d) be a complete metric space. Suppose that $T : X \to P_{\mathrm{cl,bd}}(X)$ is a contraction mapping in the sense that for some $0 \le \alpha < 1$,*

$$H(Tx, Ty) \le \alpha d(x, y), \tag{1.6}$$

for all $x, y \in X$. Then there exists a point $x \in X$ such that $x \in Tx$.

In the following theorem, Daffer and Kaneko [9] proved the existence of a fixed point for a multi-valued weak contraction mappings of a complete metric space X into $P_{\mathrm{cl,bd}}(X)$.

Theorem 1.2. *Let (X, d) be a complete metric space. Suppose that $T : X \to P_{\mathrm{cl,bd}}(X)$ is a contraction mapping in the sense that for some $0 \le \alpha < 1$,*

$$H(Tx, Ty) \le \alpha N(x, y), \tag{1.7}$$

for all $x, y \in X$ (i.e., weak contraction). If $x \mapsto d(x, Tx)$ is lower semicontinuous (l.s.c.), then there exists a point $x_0 \in X$ such that $x_0 \in Tx_0$.

Rouhani and Moradi [14] extended the Nadler and Daffer–Kaneko theorems to a coincidence theorem, without assuming $x \mapsto d(x, Tx)$ to be l.s.c.

A mapping $T : X \to P_{\mathrm{cl,bd}}(X)$ has the approximate endpoint property [8] if

$$\inf_{x\in X} \sup_{y\in Tx} d(x, y) = 0. \tag{1.8}$$

Let $T : X \to X$ be a single-valued mapping. Then T has the approximate endpoint property if and only if T has the approximate fixed point property, i.e.,

$$\inf_{x\in X} d(x, Tx) = 0. \tag{1.9}$$

In the following theorem, Amini-Harandi [8] in 2010 proved the following endpoint result for a multi-valued mappings of a complete metric space X into $P_{\mathrm{cl,bd}}(X)$.

Theorem 1.3 (*[8, Theorem 2.1]*)**.** *Let (X, d) be a complete metric space. Suppose that $T : X \to P_{\mathrm{cl,bd}}(X)$ is a multi-valued mapping that satisfies*

$$H(Tx, Ty) \le \psi(d(x, y)), \tag{1.10}$$

for each $x, y \in X$, where $\psi : [0, +\infty) \to [0, +\infty)$ is upper semicontinuous, with $\psi(t) < t$ for all $t > 0$, satisfying $\liminf_{t\to\infty}(t - \psi(t)) > 0$. Then T has a unique endpoint if and only if T has the approximate endpoint property.

In Section 2, we prove an endpoint theorem for generalized weak contractive mappings.

The mapping $T : X \to X$ ($T : X \to P_{\mathrm{cl,bd}}(X)$) is said to be a generalized weak contraction if there exists an upper semicontinuous mapping (u.s.c.) $\psi : [0, +\infty) \to [0, +\infty)$ satisfying $\psi(t) < t$ for all $t > 0$ such that

$$d(Tx, Ty) \le \psi(N(x, y)) \qquad \Big(H(Tx, Ty) \le \psi(N(x, y))\Big), \tag{1.11}$$

for all $x, y \in X$.