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A positive solution branch for nonlinear eigenvalue problems in $\mathbb{R}^N$

T.V. Anoop [a,b,*], Jagmohan Tyagi [a]

[a] *TIFR Center for Applicable Mathematics, Yelahanka, Bangalore 560065, India*

[b] *The Institute of Mathematical Sciences, Chennai 600113, India*

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ABSTRACT

In this note, we give sufficient conditions for the existence of a positive solution branch for the problem

$$\begin{aligned} &-\Delta u = \lambda a(x) f(u) \quad \text{in } \mathbb{R}^N, \\ &u(x) \to 0 \quad \text{as } |x| \to \infty, \end{aligned}$$

with sign changing weight a, where $N \geq 3$ and f is a smooth nonlinearity with $f(0) \neq 0$.

1. Introduction

Many problems in mathematical physics, for example, wave phenomena, nonlinear field theory, combustion theory, fluid dynamics etc., lead to a nonlinear eigenvalue problem of the type

$$\begin{aligned} &-\Delta u = \lambda f(u) \quad \text{in } \Omega \subset \mathbb{R}^N \\ &u(x) = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where the existence of a positive solution is of great importance. Recently, many researchers have been interested in studying the following problems in $\Omega \subseteq \mathbb{R}^N$ with weight a:

$$\begin{aligned} &-\Delta u = \lambda a(x) f(u) \quad \text{in } \Omega, \\ &u(x) = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

due to the appearance of these kinds of problems in population genetics.

For the case $f(0) = 0$, there are some works dealing with the existence of bounded positive $\mathcal{C}^2$ solutions to (1.1) in $\mathbb{R}^N$ for λ sufficiently large, when $f(1) = 0$, f is smooth and $f(s) > 0, s \in (0, 1)$. For example in [1] Tertikas proved the existence result when a is locally Hölder continuous and negative at infinity. For $N \geq 3$, Brown and Stavrakakis [2] proved the same result for when a is smooth and lies in $L^{\frac{N}{2}}$. The decay of solutions are obtained under additional assumptions on the weight a. Gámez [3] established the existence of solutions, in $\mathcal{D}_0^{1,2}(\mathbb{R}^N) :=$ the completion of $\mathcal{C}_c^{\infty}(\mathbb{R}^N)$ with respect the norm $\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{\frac{1}{2}}$, to (1.1) when a is continuous and the positive part $a^+ \in L^{\frac{2N}{N+2}}$ and he relaxed the smoothness of f to Lipschitz continuity and differentiability at 0. He proved the decay under the additional assumption that a is bounded.

For the case $f(0) > 0$, some existence results are available when Ω is bounded and λ is sufficiently small. In [4], the authors established the existence of a nonnegative radial solution to (1.1) in B_1, for sign changing a, when f is positive and

* Corresponding author at: TIFR Center for Applicable Mathematics, Yelahanka, Bangalore 560065, India. Tel.: +91 009742395175.
E-mail addresses: tvanoop@math.tifrbng.res.in, tvanoop@imsc.res.in (T.V. Anoop), tyagi@math.tifrbng.res.in, jtyagi1@gmail.com (J. Tyagi).

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nondecreasing on $[0, \infty)$. Further, they assumed that $a \in L^1(0, 1)$ and a condition on the interaction of a^+ and a^-, namely that there exists an $\epsilon > 0$ such that

$$\int_0^t x^{N-1} a^+(x)\mathrm{d}x \geq (1+\epsilon) \int_0^t x^{N-1} a^-(x)\mathrm{d}x, \quad \forall t \in [0, 1]. \tag{1.2}$$

They obtained the solution in the space of bounded, continuous functions as a fixed point of an integral operator. Cac et al. [5] generalized the result of [4] to any bounded domain with smooth boundary, relaxing the nondecreasing assumption on the positive function f. Further they demand a to be in $L^s(\Omega)$ for $s > \max\left\{1, \frac{N}{2}\right\}$ and a condition similar to their condition (1.2), that the problem

$$-\Delta w = a^+(x) - (1+\epsilon)a^-(x), \quad x \in \Omega,\ w = 0 \text{ on } \partial\Omega \tag{1.3}$$

has a nonnegative weak solution for some $\varepsilon > 0$. Hai [6] established the existence of a positive solution in any arbitrary bounded domain to the problem (1.1) in $C(\bar{\Omega})$, relaxing the positivity assumption on f. They assumed that $f(0) > 0$ and $a \in C(\bar{\Omega})$. Their condition, similar to (1.2) and (1.3), is that there exists a number $k > 1$ such that

$$\int_\Omega G(x, y)a^+(y)\mathrm{d}y \geq k \int_\Omega G(x, y)a^-(y)\mathrm{d}y, \quad \forall x \in \Omega, \tag{1.4}$$

where $G(x, y)$ is the Green's function of $-\Delta$ in Ω with Dirichlet boundary conditions.

Afrouzi and Brown [7] studied the same problem in bounded domains for smooth f and established the existence of a positive classical solution for λ small, using the implicit function theorem. They related the existence of a positive solution of the problem (1.1) to the positivity of the solution v of the linearized problem

$$\begin{aligned} -\Delta v &= f(0)a(x) \quad \text{in } \Omega, \\ v(x) &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.5}$$

Positivity of the solution of (1.5) induces the following condition on a:

$$\int_\Omega G(x, y)a^+(x)\mathrm{d}y > \int_\Omega G(x, y)a^-(x)\mathrm{d}y, \quad \forall x \in \Omega. \tag{1.6}$$

Notice that all the earlier conditions, namely (1.2)–(1.4), imply the above condition (1.6) on a via (1.5).

To the best of our knowledge, there is no result about the existence of positive solutions to the problem (1.1) in $\mathbb{R}^N$ in the case when $f(0) \neq 0$. In the present study, our aim is to establish the existence of a positive solution to (1.1) in $\mathbb{R}^N$ when $f(0) \neq 0$, in the spirit of [7], using a similar condition on the linearized problem. More precisely, we establish the existence of a positive weak solution in $D_0^{1,2}(\mathbb{R}^N),\ N \geq 3$, to the problem

$$\begin{aligned} -\Delta u &= \lambda a(x) f(u) \quad \text{in } \mathbb{R}^N, \\ u(x) &\to 0 \quad \text{as } |x| \to \infty, \end{aligned} \tag{1.7}$$

where λ is a positive parameter, i.e., $u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N)$ such that $u > 0$ a.e. and

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v \,\mathrm{d}z = \lambda \int_{\mathbb{R}^N} a(x) f(u) v \,\mathrm{d}x, \quad \forall v \in \mathcal{D}_0^{1,2}(\mathbb{R}^N). \tag{1.8}$$

For further studies, we assume the following:

(H1) $f \in \mathcal{C}^1(\mathbb{R})$ such that $f(0) \neq 0$ and there exists $s_0 > 0$ such that

$$|f'(s)| \leq C|s|^{\gamma-2}, \quad \forall |s| \geq s_0,$$

with $\gamma \in [2, 2^*)$, where $2^* = \frac{2N}{N-2}$.

(H2) $a \in L^{\frac{2N}{N+2}}(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ with $r > \widetilde{p}$, where $\widetilde{p}$ is the conjugate of $\left(\frac{2^*}{\gamma}\right)$.

(H3) $f(0)a(x) \geq 0$ a.e. near infinity.

The main novelty of our hypotheses is that the weight function a need not be smooth, and the function f is not necessarily bounded, but we demand that a lies in certain Lebesgue spaces and f is $\mathcal{C}^1(\mathbb{R})$ with subcritical growth at infinity. Now we state the main results:

Theorem 1.1. *Let* (H1)–(H3) *hold and let* $v \in D_0^{1,2}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$ *be a weak solution of*

$$-\Delta v = f(0)a(x) \quad \text{in } \mathbb{R}^N. \tag{1.9}$$

If $v > 0$ *in* $\mathbb{R}^N$, *then there exists a* $\lambda_0 > 0$ *such that* (1.7) *has a positive weak solution* $u_\lambda \in D_0^{1,2}(\mathbb{R}^N)$ *for* $0 < \lambda < \lambda_0$.