Nonlinear Analysis 74 (2011) 2201–2209

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Fixed point results for mappings satisfying (ψ, φ)-weakly contractive condition in partially ordered metric spaces

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ARTICLE INFO

Article history:
Received 22 July 2010
Accepted 11 November 2010

MSC:
41A50
47H10
54H25

Keywords:
Fixed point
Coincidence point
Common fixed point
Altering distance function
Weakly contractive condition
Weakly increasing mappings
Partially ordered set

ABSTRACT

We establish coincidence fixed point and common fixed point theorems for mappings satisfying (ψ, φ)-weakly contractive condition in an ordered complete metric space. Some applications of our obtained results are given.

1. Introduction

The Banach contraction principle [1] is a very popular tool for solving existence problems in many branches of mathematical analysis. This famous theorem can be stated as follows.

Theorem 1.1 *([1]). Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T}$ be a mapping of $\mathcal{X}$ into itself satisfying:*

$$d(\mathcal{T}x, \mathcal{T}y) \leq kd(x, y), \quad \forall x, y \in X, \tag{1.1}$$

where k is a some constant in $(0, 1)$. Then, $\mathcal{T}$ has a unique fixed point $x^* \in \mathcal{X}$.

In the literature there is a large number of generalizations of the Banach contraction principle (see [2–37] and others). In particular, obtaining the existence and uniqueness of fixed points for self-maps on a metric space by altering distances between the points with the use of a certain control function is an interesting aspect. There are control functions which alter the distance between two points in a metric space. In this direction, Khan et al. [22] addressed a new category of fixed point problems for a single self-map with the help of a control function which they called an altering distance function.

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0362-546X/$ – see front matter
doi:10.1016/j.na.2010.11.024

Definition 1.2 (*Altering Distance Function [22]*). $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

(a) φ is continuous and non-decreasing,
(b) $\varphi(t) = 0 \Leftrightarrow t = 0$.

Theorem 1.3 (*[22]*). *Let $(\mathcal{X}, d)$ be a complete metric space, let φ be an altering distance function, and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a self-mapping which satisfies the following inequality:*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq c\varphi(d(x, y)) \tag{1.2}$$

for all $x, y \in \mathcal{X}$ and for some $0 < c < 1$. Then, $\mathcal{T}$ has a unique fixed point.

Putting $\varphi(t) = t$ in the previous theorem, (1.2) reduces to (1.1).

In [3], Alber and Guerre-Delabriere introduced the concept of weak contraction in Hilbert spaces. Rhoades [30] has shown that the result which Alber et al. had proved in [3] is also valid in complete metric spaces.

Definition 1.4 (*Weakly Contractive Mapping*). Let $\mathcal{X}$ be a metric space. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is called weakly contractive if and only if:

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(x, y) - \varphi(d(x, y)), \quad \forall x, y \in \mathcal{X}, \tag{1.3}$$

where φ is an altering distance function.

Theorem 1.5 (*[30, Theorem 2]*). *Let $(\mathcal{X}, d)$ be a complete metric space. If $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a weakly contractive mapping, then $\mathcal{T}$ has a unique fixed point.*

Note that Alber et al. [3] assumed an additional condition on φ which is $\lim_{t\to+\infty} \varphi(t) = +\infty$. But Rhoades [30] obtained the result noted in Theorem 1.5 without using this particular assumption. If one takes $\varphi(t) = (1 - k)t$, where $0 < k < 1$, then (1.3) reduces to (1.1).

Dutta and Choudhury in [16] obtained the following generalization of Theorems 1.3 and 1.5.

Theorem 1.6 (*[16]*). *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfying:*

$$\psi(d(\mathcal{T}x, \mathcal{T}y)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \tag{1.4}$$

for all $x, y \in \mathcal{X}$, where ψ and φ are altering distance functions. Then $\mathcal{T}$ has a unique fixed point.

Weak inequalities of the above type have been used to establish fixed point results in a number of subsequent works, some of which are noted in [6,9,35].

In recent years, many results appeared related to fixed point theorems in a complete ordered metric space. The first result in this direction was given by Ran and Reurings [27, Theorem 2.1], where they extended the Banach contraction principle [1] in partially ordered sets with some applications to linear and nonlinear matrix equations. Subsequently, Nieto and Rodríguez-López [24] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Similar applications based on a version of Theorems 2.1–2.5 [24] for a mixed monotone mapping $\mathcal{F} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ were given by Bhaskar and Lakshmikantham [7]. Further improvements of the above discussed results were found independently in [2,4,5,10,12,15,21,23,25,26,28,31,33,34,36,37].

Further, in the year 2009, Harjani and Sadarangani [17] used the above discussed concept and proved some fixed point theorems for weakly contractive operators in ordered metric spaces. Subsequently, Harjani and Sadarangani [18] generalized their own results [17] by considering a pair of altering functions (ψ, φ).

In this paper, by considering a pair of altering functions (ψ, φ), we establish coincidence point and common fixed point theorems for mappings satisfying a generalized weakly contractive condition in an ordered complete metric space. As applications, we give some fixed point theorems for contractions of integral type and an existence theorem for a solution of an integral equation.

2. Main results

At first, we introduce some notations and definitions that will be used later.

2.1. Notations and definitions

The following definition was introduced by Jungck in [20].

Definition 2.1 (*[20]*). Let $(\mathcal{X}, d)$ be a metric space and $f, g : \mathcal{X} \to \mathcal{X}$. If $w = fx = gx$, for some $x \in \mathcal{X}$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. The pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n\to+\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some $t \in \mathcal{X}$.