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Robust conjugate duality for convex optimization under uncertainty with application to data classification[☆]

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ABSTRACT

In this paper we present a robust conjugate duality theory for convex programming problems in the face of data uncertainty within the framework of robust optimization, extending the powerful conjugate duality technique. We first establish robust strong duality between an uncertain primal parameterized convex programming model problem and its uncertain conjugate dual by proving strong duality between the deterministic robust counterpart of the primal model and the optimistic counterpart of its dual problem under a regularity condition. This regularity condition is not only sufficient for robust duality but also necessary for it whenever robust duality holds for every linear perturbation of the objective function of the primal model problem. More importantly, we show that robust strong duality always holds for partially finite convex programming problems under scenario data uncertainty and that the optimistic counterpart of the dual is a tractable finite dimensional problem. As an application, we also derive a robust conjugate duality theorem for support vector machines which are a class of important convex optimization models for classifying two labelled data sets. The support vector machine has emerged as a powerful modelling tool for machine learning problems of data classification that arise in many areas of application in information and computer sciences.

1. Introduction

Duality theory is a cornerstone in the area of constrained optimization and has been studied for over a century. However, real-world problems of constrained optimization often involve input data that are noisy or uncertain due to modelling or measurement errors [1–3]. Consequently, how to develop mathematical approaches that are capable of treating data uncertainty in constrained optimization has become a critical question in mathematical optimization. Over the years, various deterministic as well as stochastic approaches have been developed for treating uncertainty in optimization (see [4–10] and other references therein). In this paper, we examine a robust optimization framework [1] for studying conjugate duality theory for constrained optimization in the face of data uncertainty.

Consider the standard form convex optimization problem in the absence of data uncertainty

$$(P)\quad \inf_{x\in X} f(x),$$

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where $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function. This problem can be embedded into a family of parameterized problems (see [11])

$$(P_y)\ \inf_{x \in X} \phi(x, y),$$

where the function $\phi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ satisfies $\phi(x, 0) = f(x)$. Clearly, (P_0) collapses to the original problem (P). The parameterized convex optimization problem (P_y) in the face of data uncertainty can be captured by the problem

$$(P_{y,u})\ \inf_{x \in X} \phi_u(x, y),$$

where $\phi_u : X \times Y \to \mathbb{R}$ is a proper lower semicontinuous convex function and u is the uncertain parameter which belongs to the uncertainty set $\mathcal{U}$. For instance, the effect of uncertain data (a_1, a_2) on the constraint of the problem

$$\min\{h(x) \mid a_1x_1 + a_2x_2 \le b\}$$

can be captured by the problem $(P_{0,u})$, $\inf_{x \in X} \phi_u(x, 0)$, where

$$\phi_u(x, y) = h(x) + \delta_{\{z:\ a_1(u_1)z_1 + a_2(u_2)z_2 \le b+y\}}(x),$$

where $\delta_C(.)$ denotes the indicator function of a set C and the parameter $u = (u_1, u_2)$ is in an interval uncertainty set $\mathcal{U} = [c_1, d_1] \times [c_2, d_2]$.

The **robust counterpart** (RP) of problem $(P_{0,u})$ is the deterministic optimization problem

$$\inf_{x \in X} \sup_{u \in \mathcal{U}} \phi_u(x, 0).$$

On the other hand, for each fixed $u \in \mathcal{U}$, the conjugate dual problem of $(P_{0,u})$ is given by

$$\max_{y^* \in Y^*} \{-\phi_u^*(0, y^*)\}.$$

The **optimistic counterpart** (ODP) of the uncertain dual problem is also a deterministic optimization problem which is given by

$$\max_{u \in \mathcal{U}} \max_{y^* \in Y^*} \{-\phi_u^*(0, y^*)\}.$$

We say that **robust strong duality** holds whenever the values of the robust counterpart and the optimistic counterpart coincide with the dual attainment, i.e.,

$$\inf_{x \in X} \sup_{u \in \mathcal{U}} \phi_u(x, 0) = \max_{u \in \mathcal{U}} \max_{y^* \in Y^*} \{-\phi_u^*(0, y^*)\}.$$

In this paper, we first establish robust strong duality under the condition that

$$\mathrm{Pr}_{X^* \times \mathbb{R}} \left(\bigcup_{u \in \mathcal{U}} \mathrm{epi}\phi_u^* \right) \text{ is } w^*\text{-closed and convex.} \tag{1.1}$$

We then show that this condition is not only sufficient for robust duality but also is necessary for robust strong duality to hold for every linear perturbation of $\phi_u(x, y)$ in the sense that, for each linear functional x^*,

$$\inf_{x \in X} \sup_{u \in \mathcal{U}} \{\phi_u(x, 0) + \langle x^*, x \rangle\} = \max_{u \in \mathcal{U}} \max_{y^* \in Y^*} \{-\phi_u^*(-x^*, y^*)\}.$$

For related recent conjugate duality results without data uncertainties, see [12–17]. A more recent exhaustive treatment of conjugate duality in the absence of data uncertainty can be found in [18].

We also prove that robust strong duality always holds for partially finite convex programming problems under scenario data uncertainty [1] by verifying (1.1). In this case we also see that the optimistic counterpart of the dual is a tractable finite dimensional convex problem. Partially finite convex programs cover broad classes of problems including constrained approximation problems and interpolation problems. For related results for partially finite convex programs, see [19,20].

As an application, we derive a robust Fenchel duality theorem for support vector machines [21–23] which are a class of convex optimization models for classifying two labelled data sets. The support vector machine has emerged as a powerful modelling tool for machine learning problems of data classification that arise in many areas of information and computer sciences [24–27].

The outline of the paper is as follows. Section 2 presents preliminaries of convex analysis that will be used later in the paper. Section 3 develops robust conjugate duality theorems for convex optimization including cone-constrained optimization under uncertainty. Section 4 shows that a robust conjugate duality theorem always holds for partially finite convex programs under scenario uncertainty and illustrates that these problems are tractable computationally. Section 5 derives a robust Fenchel duality under uncertainty, extending the classical Fenchel duality. Section 6 provides an application of robust Fenchel duality to a convex optimization model that arises in data classification problems. Finally, we present additional regularity conditions ensuring robust conjugate duality in the Appendix.