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Variational sets: Calculus and applications to nonsmooth vector optimization

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ABSTRACT

We develop elements of calculus of variational sets for set-valued mappings, which were recently introduced in Khanh and Tuan (2008) [1,2] to replace generalized derivatives in establishing optimality conditions in nonsmooth optimization. Most of the usual calculus rules, from chain and sum rules to rules for unions, intersections, products and other operations on mappings, are established. Direct applications in stability and optimality conditions for various vector optimization problems are provided.

1. Introduction

In nonsmooth optimization, many generalized derivatives have been introduced to replace the Fréchet and Gateaux derivatives which do not exist. Each of them is adequate for some classes of problems, but not all. In [1,2] we proposed two kinds of variational sets for mappings between normed spaces. These subsets of the image space are larger than the images of the pre-image space through known generalized set-valued mappings. Hence our necessary optimality conditions obtained by separation techniques are stronger than many known conditions using various generalized derivatives. Of course, sufficient optimality conditions based on separations of bigger sets may be weaker. But in [1,2], using variational sets we can establish sufficient conditions which have almost no gap with the corresponding necessary ones. The second advantage of the variational sets is that we can define these sets of any order to get higher-order optimality conditions. This feature is significant since many important and powerful generalized derivatives can be defined only for the first and second orders and the higher-order optimality conditions available in the literature are much fewer than the first and second-order ones. The third strong point of the variational sets is that almost no assumptions are needed to be imposed for their being well-defined and nonempty and also for establishing optimality conditions. Calculating them from the definition is only a computation of a Painlevé–Kuratowski limit. However, in [1,2] no calculus rules for variational sets are provided.

In the present paper we establish elements of calculus for variational sets to ensure that they can be used in practice. Most of the usual rules, from the sum and chain rules to various operations in analysis, are investigated. It turns out that the

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variational sets possess many fundamental and comprehensive calculus rules. Although this construction is not comparable with objects in the dual approach like Mordukhovich's coderivatives (see the excellent books [3,4]) in enjoying rich calculus, it may be better in dealing with higher-order properties. We pay attentions also on relations between the established calculus rules and applications of some rules to get others. Of course, significant applications should be those in other topics of nonlinear analysis and optimization. As such applications we provide a direct employment of sum rules to establishing an explicit formula for a variational set of the solution map to a parameterized variational inequality in terms of variational sets of the data. Furthermore, chain rules and sum and product rules are also used to prove optimality conditions for weak solutions of some vector optimization problems.

The organization of the paper is as follows. The rest of this section is devoted to recalling definitions needed in the sequel. We present the two kinds of higher-order variational sets, including various equivalent formulations and simple properties in Section 2. In the next Section 3 we explore comprehensive calculus rules for the variational sets. We also try to illustrate by example the unfortunate lack of expected rules. We present in Section 4 direct applications of chain rules and sum and product rules obtained in Section 3 to considering stability and optimality conditions, as mentioned above.

Throughout the paper, if not otherwise specified, let X and Y be real normed spaces, $C \subseteq Y$ a closed pointed convex cone with nonempty interior and $F : X \to 2^Y$. For $A \subseteq X$, int$A$, cl$A$ (or $\bar{A}$), bd$A$ denote its interior, closure and boundary, respectively. X^* is the dual space of X and B_X stands for the closed unit ball in X. For $x_0 \in X$, $U(x_0)$ is used for the set of all neighborhoods of $x_0 \in X$. $\mathbb{R}^k_+$ is the nonnegative orthant of the k-dimensional space. $\mathbb{N}$ stands for the set of natural numbers. For $r > 0$ tending to 0, $0(r)$ and $\vartheta(r)$ mean a moving point z in the space in question (always clear from the context) such that $\frac{1}{r}\|z\| \to 0$ and $\|z\| \to 0$, respectively. We often use the following cones, for $A \subseteq X$, C above and $u \in X$,

$$\begin{aligned}
&\text{cone}A = \{\lambda a \mid \lambda \geq 0,\ a \in A\},\\
&\text{cone}_+A = \{\lambda a \mid \lambda > 0,\ a \in A\},\\
&A(u) = \text{cone}(A + u),\\
&C^* = \{y^* \in Y^* \mid \langle y^*, c\rangle \geq 0,\ \forall c \in C\}\ (\text{polar cone}),\\
&C^\sharp = \{y^* \in Y^* \mid \langle y^*, c\rangle > 0,\ \forall c \in C \setminus \{0\}\}\ (\text{quasi interior of } C^*).
\end{aligned}$$

A nonempty convex subset B of a convex cone C is called a base of C if $C = \text{cone}B$ and $0 \notin \text{cl}B$.

For a set-valued map (known in the literature also as multimap or point-to-set map or multifunction or correspondence) $H : X \to 2^Y$, the domain, graph and epigraph of H are defined as

$$\begin{aligned}
&\text{dom}H = \{x \in X : H(x) \neq \emptyset\}, \qquad \text{gr}H = \{(x, y) \in X \times Y : y \in H(x)\},\\
&\text{epi}H = \{(x, y) \in X \times Y : y \in H(x) + C\}.
\end{aligned}$$

The so-called profile mapping of H is H_+ defined by $H_+(x) = H(x) + C$. The Painlevé–Kuratowski (sequential) outer (or upper) limit is defined by

$$\operatorname*{Limsup}_{x \xrightarrow{H} x_0} H(x) = \{y \in Y \mid \exists x_n \in \text{dom}H :\ x_n \to x_0, \exists y_n \in H(x_n),\ y_n \to y\},$$

where $x \xrightarrow{H} x_0$ means that $x_n \in \text{dom}H$ and $x_n \to x_0$. The Painlevé–Kuratowski lower limit is

$$\operatorname*{Liminf}_{x \xrightarrow{H} x_0} H(x) = \{y \in Y \mid \forall x_n \in \text{dom}H :\ x_n \to x_0, \exists y_n \in H(x_n),\ y_n \to y\}.$$

H is said to be compact at $x_0 \in \text{cl}(\text{dom}H)$ if any sequence $(x_n, y_n) \in \text{gr}H$ has a convergent subsequence as soon as $x_n \to x_0$. The closure of H is the multimap cl$H$, whose graph is defined as the closure of gr$H$. Thus

$$\text{cl}H(x_0) = \operatorname*{Limsup}_{x \xrightarrow{H} x_0} H(x).$$

When $\text{cl}H(x_0) = H(x_0)$ we say that H is closed at x_0.

For a subset $A \subseteq X$, the contingent cone of A at $x_0 \in X$ is

$$T_A(x_0) = \{u \in X \mid \exists t_n \to 0^+, \exists u_n \to u,\ \forall n, x_0 + t_n u_n \in A\}.$$

Assume that $u_1, \ldots, u_{m-1} \in X$ and $m \in \mathbb{N}$. The mth-order contingent set of A at $(x_0, u_1, \ldots, u_{m-1})$ is

$$T^m_A(x_0, u_1, \ldots, u_{m-1}) = \operatorname*{Limsup}_{t \to 0^+} \frac{1}{t^m}(A - x_0 - tu_1 - \cdots - t^{m-1}u_{m-1}).$$

The mth-order adjacent set of A at $(x_0, u_1, \ldots, u_{m-1})$ is

$$T^{bm}_A(x_0, u_1, \ldots, u_{m-1}) = \operatorname*{Liminf}_{t \to 0^+} \frac{1}{t^m}(A - x_0 - tu_1 - \cdots - t^{m-1}u_{m-1}).$$