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Normal cones to infinite intersections

Thomas I. Seidman *

Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, MD 21250, USA

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ABSTRACT

For sets given as finite intersections $\mathcal{A} = \bigcap_{k=1}^{K} \mathcal{A}_k$ the basic normal cone $N(\bar{x}; \mathcal{A})$ is given as $\sum_k N(\bar{x}; \mathcal{A}_k)$, but such a result is not, in general, available for infinite intersections. A comparable characterization of $N(\bar{x}; \mathcal{A})$ is obtained here for a class of such infinite intersections.

1. Introduction

A fundamental concern in variational analysis is the characterization through first-order optimality conditions of the solution of a constrained minimization problem

$$\text{Minimize: } f(x) \quad \text{subject to: } x \in \mathcal{A}. \tag{1}$$

Under quite general conditions the first-order conditions on a local minimizer $\bar{x}$ are known [1, Prop. 5.1] to take the form

$$-[\nabla f](\bar{x}) \in \hat{N}_0(\bar{x}; \mathcal{A}) \subset N(\bar{x}; \mathcal{A}) \tag{2}$$

where $N(\bar{x}; \mathcal{A})$ denotes the basic normal cone at $\bar{x}$ ([2]; see Definitions 2.1 and 2.2) to the admissible set $\mathcal{A}$. Thus we are led to the task of computing $N(\bar{x}; \mathcal{A})$ from the specification provided for $\mathcal{A}$.

Constraints in large-scale problems are often generated by replicating a limited number of constraint prototypes over large index sets. In particular, we may consider, as prototype, constraints of the form $\varphi \geq 0$ for a scalar function φ, noting that the constraints in problem (1) are often given in the form of a family of such inequalities

$$\text{Minimize: } f(x) \quad \text{subject to: } \varphi(x) \geq 0 \quad \text{for } \varphi \in \Phi \tag{3}$$

where Φ is a set of constraint functions $\varphi : \mathcal{X} \to \mathbb{R}$. [While this set of inequalities could be subsumed by a single inequality $\varphi_*(x) \geq 0$ on taking $\varphi_*(x) = \inf\{\varphi(x) : \varphi \in \Phi\}$: this does not seem helpful. Indeed, we would have $-\mathrm{epi}\varphi_* = \bigcap_{\varphi\in\Phi}\{-\mathrm{epi}(\varphi)\}$ so the present consideration of infinite intersections seems as likely to help with differentiation of sup or inf as the reverse situation. We also note that (3) is frequently seen with the inequality reversed, simply corresponding to the replacement $\varphi \leftarrow -\varphi$.]

Note that (3) means that the admissible set $\mathcal{A}$ is presented as an intersection

$$\mathcal{A} = \bigcap_{\varphi\in\Phi} \mathcal{A}^{\varphi} \quad \text{with } \mathcal{A}^{\varphi} = \{x : \varphi(x) \geq 0\} = \varphi^{-1}([0, \infty)) \tag{4}$$

* Tel.: +1 410 455 2438; fax: +1 410 455 1066.
E-mail address: seidman@math.umbc.edu.

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and we would like to determine $N(\bar{x}; \mathcal{A})$ in terms of the functions $\varphi \in \Phi$. When this is a finite intersection ($\Phi = \{\varphi_1, \ldots, \varphi_K\}$), one has, under quite mild conditions, an 'intersection rule'

$$N\left(\bar{x}; \bigcap_{\varphi\in\Phi} \mathcal{A}^{\varphi}\right) = \sum_{\varphi\in\Phi} N(\bar{x}; \mathcal{A}^{\varphi}) \tag{5}$$

as a special case of [2, Cor. 3.5]. In general, however, no such results are available for infinite intersections. [We do note that (5) is somewhat analogous to differentiating a sum, so working with an infinite intersection might be compared to term-by-term differentiation of a series; one expects this to be possible, but under more restrictive hypotheses and perhaps with a modified statement. Certainly we would expect this shift to involve some new ideas — e.g., there is an interchange of limits in the background so we might expect to require some uniformity condition. Even under hypotheses ensuring its validity for each finite subset of Φ, it is clear that the formula (5) will generally be false, as stated, for infinite Φ.]

Example 1.1. We might consider (4) in a restricted but typical setting: for example $\mathcal{A}$ might be the set of non-negative functions in, for example, $\mathcal{X} = C(\Omega)$. This is, indeed, of the form (4) we are considering, here with Φ the set of evaluation functionals $\Phi = \{[x(\cdot) \mapsto x(s)] : s \in \Omega\}$, noting that this is rather special in that $\mathcal{A}$ is here a closed convex cone in $\mathcal{X}$.

Somewhat more generally, we might consider an arbitrary closed convex set in a Banach space $\mathcal{X}$ and note that this is describable as the intersection of all the half-spaces containing it:

$$\mathcal{A} = \bigcap_{\alpha\in\mathcal{I}} \mathcal{A}_{\alpha} \qquad \mathcal{A}_{\alpha} = \{x : \langle \xi_{\alpha}, x\rangle \geq \rho_{\alpha}\} \tag{6}$$

which is of the form (4) with $\varphi \in \Phi$ of the form $\varphi(x) = \langle \xi, x\rangle - \rho$. Given $\bar{x}$ at the boundary of $\mathcal{A}$, the *active set* of constraints is given by $\Phi_* = \{\varphi \in \Phi : \langle \xi, \bar{x}\rangle = \rho\}$ so the set of support functionals at $\bar{x}$ is $\{-\xi : \varphi \in \Phi_*\}$. Since $N(\bar{x}; \mathcal{A}_{\alpha}) = \{0\}$ for the inactive constraints $\alpha \in \mathcal{I} \setminus \mathcal{I}_*$, we have

$$N(\bar{x}; \mathcal{A}) = \overline{\mathrm{co}}\,\{N(\bar{x}; \mathcal{A}_{\alpha}) : \alpha \in \mathcal{I}\} \tag{7}$$

where "$\overline{\mathrm{co}}\,\mathcal{S}$" denotes the conical hull of $\mathcal{S}$, i.e., the closure of the convex hull of $\{a\xi : a > 0, \xi \in \mathcal{S}\}$, so (7) is the natural interpretation of (5). This characterization depends on our having used the complete set of support functionals in specifying $\mathcal{A}$.

Example 1.2. As another simple example, take $\mathcal{X} = \mathbb{R}$, $\bar{x} = 0$, and let $\varphi_k(x) = x + 1/k$. This gives $0 = \bar{x}$ in the interior of each $\mathcal{A}^{\varphi_k} = [-1/k, \infty)$ so no given constraint would be active. Here, each $N(0; \mathcal{A}^{\varphi_k}) = \{0\}$, while $\mathcal{A} = [0, \infty)$ so $N(0; \mathcal{A}) = (-\infty, 0] \neq \{0\}$.

Example 1.3. A slightly different example takes $\mathcal{A}$ to be the unit disk centered at the origin of $\mathbb{R}^2$ which we present as the infinite intersection of the countable set of half-spaces $\mathcal{A}^{\varphi}$ given by

$$\varphi_{\pm k}(x) = 1 - \left(r_k, \pm\sqrt{1 - r_k^2}\right) \cdot x$$

where (r_k) is an enumeration of the rationals in $[-1, 1]$. If we then take $\bar{x} = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$, none of these constraints are active since $\bar{x}$ is in the interior of each of the presenting half-spaces $\mathcal{A}^{\varphi_k}$ above and the support functionals (a, a) exactly at $\bar{x}$ do not appear in the specifying $\{\varphi_k\}$. Nevertheless, the normal cone $N(\bar{x}; \mathcal{A}) = \{(-a, -a) : a \geq 0\}$ is expressible in terms of these through the neighboring support functionals ($\varphi_k(\bar{x}) \approx 0$):

$$N(\bar{x}; \mathcal{A}) = -\bigcap_{\omega>0} \overline{\mathrm{co}}\,\{\varphi'(\bar{x}) : \varphi_k(\bar{x}) < \omega\} \tag{8}$$

noting that $\varphi_k(\bar{x}) \leq \omega$ gives $a\varphi_k'$ in a wedge centered at (a, a) of angular width diminishing to 0 as $\omega \to 0$.

Example 1.4. Again with $\mathcal{X} = \mathbb{R}^2$, $\bar{x} = (0, 0)$, one might consider a quite different variant taking $\varphi_k(x, y) = \{x \text{ if } y \geq 0;\ x - ky^2 \text{ if } y \leq 0\}$. Here each constraint is active at $\bar{x}$ and each $N(\bar{x}; \mathcal{A}^{\varphi_k}) = \{(-r, 0) : r \geq 0\}$. However, $\mathcal{A} = \{(x, y) : x, y \geq 0\}$ with $N(\bar{x}; \mathcal{A}) = \{(r, s) : r, s \leq 0\}$ and this is not contained in the hull of $\{N(\bar{x}; \mathcal{A}^{\varphi_k}) : k = 1, 2, \ldots\}$.

It is clear from Example 1.3 that the active constraints taken from a set of specifying constraints need not be sufficient and we must also consider 'almost active' constraints, for which $\varphi(\bar{x})/\|\xi\|$ is small although not exactly 0. We thus have a simultaneous concern for two questions:

1. The defining functionals are nonlinear, although moderately smooth, so the set $\mathcal{A}$ need not be convex — and, of course, there may be infinitely many such defining constraint functionals.
2. We are concerned that the set of 'presenting functionals' defining $\mathcal{A}$ may not be complete: the active constraints (support functionals at $\bar{x}$) might not appear at all.

Our goal here is to obtain a more general (nonconvex) version of the convex case (7), replacing the affine functionals $\psi = \psi^{\rho,\xi}$ appearing there by an infinite set Φ of nonlinear Fréchet differentiable functionals $\varphi : \mathcal{X} \to \mathbb{R}$ giving (4) while taking account of 'almost active' constraints in formulating the result. [We also note in this a need for some uniformity condition to rule out consideration of Example 1.4.]