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On source-type solutions and the Cauchy problem for a doubly degenerate sixth-order thin film equation, I: Local oscillatory properties

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ABSTRACT

As a key example, the sixth-order doubly degenerate parabolic equation from thin film theory,

$$u_t = (|u|^m |u_{xxxxx}|^n u_{xxxxx})_x \quad \text{in } \mathbb{R} \times \mathbb{R}_+,$$

with two parameters, $n \geq 0$ and $m \in (-n, n+2)$, is considered. In this first part of the research, various local properties of its particular travelling wave and source-type solutions are studied. Most complete analytic results on oscillatory structures of these solutions of changing sign are obtained for $m = 1$ by an algebraic–geometric approach, with extension by continuity for $m \approx 1$.

1. Introduction: Basic nonlinear model with complicated local and global properties of solutions

1.1. Higher-order degenerate parabolic PDEs: No potential, monotone, order-preserving properties, not of divergence form, and no weak solutions

We consider the sixth-order parabolic equation from thin film theory,

$$u_t = \left(|u|^m |u_{xxxxx}|^n u_{xxxxx}\right)_x \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

with two parameters, $n \geq 0$ and $m \in (-n, n+2)$. This equation is *doubly degenerate* and contains the higher-order p-Laplacian nonlinearity $|u_{xxxxx}|^n$ (then $p = 2 + n$), and another one $|u|^m$ of the porous medium type. The equation is written for solutions of changing sign, which is an intrinsic feature of the Cauchy problem (the CP) with bounded compactly supported initial data $u_0(x)$ to be studied.

The PDEs such as (1.1), which are called sixth-order *thin film equations* (the TFEs-6), were introduced by King [1] in 2001 among others for modelling of power-law fluids spreading on a horizontal substrate. Eq. (1.1) is *quasilinear*, where the diffusion-like operator includes two parameters m and n and *is not potential* (variational) and/or *monotone* in any functional setting and topology. It is not also an *operator of fully divergence form*, since the PDE admits just a single integration by parts, so that a standard definition of weak solutions is entirely illusive. Of course, as a higher-order parabolic equation, (1.1) does not exhibit any *order-preserving* (via the Maximum Principle) features.

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Overall, the TFE-6 (1.1), as a typical example of a variety of complicated nonlinear thin film models arising from modern applications, represents a serious challenge to general PDE theory of the twenty-first century, concerning principles and concepts of understanding the common local and global features and properties of its solutions, which also need proper definitions. For other sixth-order TFEs including their derivation and mathematical properties, see a survey in [2], where further key references are traced out.

It is well known that *non-negative solutions* of a wide class of higher-order TFEs can be obtained by special non-analytic and often "singular" ε-regularizations and passing to the limit $\varepsilon \to 0^+$, that, in general, lead to free-boundary problems (FBPs). We refer to the pioneering work by Bernis and Friedman [3] and to the monograph on nonlinear parabolic PDEs [4, Ch. 4], where further references and results can be found. Actually, such singular ε-regularizations, as $\varepsilon \to 0^+$, pose a kind of an "obstacle FBP", where the solutions are obliged to be non-negative by special free-boundary conditions that, in general, are not easy to detect rigorously. Solutions of the CP cannot be obtained by such techniques and require more involved and different analysis. In particular, *analytic* ε-regularizations, with $\varepsilon \to 0^+$, can be key for the CP, [5,2].

There is a large amount of pure, applied, and numerical mathematical literature devoted to existence, uniqueness, and various local and asymptotic properties of TFEs, especially, for the standard TFE-4:

$$u_t = -\big(|u|^n u_{xxx}\big)_x, \quad \text{where } n > 0. \tag{1.2}$$

Necessary key references on various results of modern TFE theory that are important for justifying principal regularity and other assumptions on solutions will be presented below and, in particular, are available in [6,7] and in a more recent paper [8]. See also [9, Ch. 3], where further references are given and several evolution properties of TFEs (with absorption, included) are discussed. However, even for simpler pure TFEs such as (1.2), questions of local and global properties of solutions of the CP, their oscillatory, and asymptotic behaviour are not completely well understood or proved in view of growing complexity of mathematics corresponding to *higher-order* degenerate parabolic flows.

Using this, rather complicated, and even exotic, model equation (1.1), we plan to explain typical and unavoidable difficulties that appear even in the study of local properties of compactly supported solutions and their interfaces for higher-order degenerate nonlinear PDEs. We then intent to give insight and develop some general approaches, notions, and techniques, that are adequate and can be applied to a wide class of difficult degenerate parabolic (and not only parabolic) PDEs. Since even the related ODEs of the fifth order for particular solutions get very complicated with a higher-dimensional phase space, we cannot rely on traditional ODE methods, which were very successful in the twentieth century for second-order ODEs, occurred for many particular self-similar and other solutions, with clear phase planes.

Instead, as a general idea, we propose to use parameter homotopy continuity approaches using the fact that for some values of m, n (e.g., for $m = 1$ or $m = n = 0$, etc.), the ODEs can be solved by some algebraic–geometric methods, or leads to easier linear equations. Then, we use a stable "transversality geometric" structure of the obtained solutions to extend those into some surrounding parameter ranges. However, global extension of those solutions are not straightforward at all and often we are obliged to apply careful numerical methods to trace out some solution properties and their actual existence.

Therefore, we are not restricted to *quasilinear* equations with semi-divergent operators. Without essential changes and hesitation, we may consider other *fully nonlinear* models such as the following formal parabolic PDE:

$$|u_t|^\sigma u_t = \big(|u|^m |u_{xxxxx}|^n u_{xxxxx}\big)_x \quad \text{in } \mathbb{R} \times \mathbb{R}_+ \ (\sigma > 0), \tag{1.3}$$

where $\sigma = 0$ leads to the quasilinear counterpart (1.1). It is easy to propose other more artificial (and often awkward) versions of such PDEs without traces or remnants of monotone, potential, divergence form, etc., operators, which however can be used for applying our general mathematical concepts of analysis.

1.2. On other models, results, and extensions

In the present first part, in Sections 2–7, we present a detailed study of some, mainly local oscillatory and sign-changing, properties of *travelling wave* (TW) and *source-type solutions*. In this connection, let us mention the first important pioneering results on oscillatory source-type solutions of the fourth-order quasilinear parabolic equation of porous medium type (the PME–4) in the fully divergence form with the monotone operator in H^{-2}:

$$u_t = -(|u|^{m-1} u)_{xxxx} \quad \text{in } \mathbb{R} \times \mathbb{R}_+ \ (m > 1), \tag{1.4}$$

which were obtained by Bernis [10] and in Bernis–McLeod [11]; see also [4, Section 4.2] for further details, and [12] for construction of a countable family of similarity solutions of (1.4).

Notice that many key features of a local oscillatory structure of solutions of quasilinear degenerate PDEs with such operators do not essentially change not only for equations of higher, 2mth order, but also for similar odd-order *nonlinear dispersion equations* (NDEs). In [13], as an illustration, we briefly review our approaches for the corresponding fifth-order counterpart of (1.1), which has the form (the NDE-5)

$$u_t = \big(|u|^m |u_{xxxx}|^n u_{xxxx}\big)_x \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \tag{1.5}$$

Various odd-order PDEs occur in nonlinear dispersion theory. As a key feature, they exhibit finite interfaces, compacton behaviour, and shock/rarefaction waves; see first results in Rosenau–Hyman [14], a survey in [9, Ch. 4], and [15], as a more