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Differential equations with non-absolutely integrable functions in ordered Banach spaces

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ABSTRACT

In this paper we derive existence and comparison results for initial value problems in ordered Banach spaces. The considered problems can be implicit, singular, functional, discontinuous and nonlocal. The main tools are fixed point results in ordered spaces and theory of HL integrable vector-valued functions. Concrete examples are presented and solved.

1. Introduction

In this paper we apply a theory of Henstock–Lebesgue (HL) integrable of vector-valued functions and fixed point results for mappings in partially ordered function spaces to derive existence and comparison results for the smallest and greatest solutions of first order initial value problems in an ordered Banach space E whose order cone is regular. The corresponding problems are studied in [1] when E is a lattice-ordered Banach space, and in [2], where improper integrals are used. A novel feature in our study is that the right-hand sides of differential equations comprise locally HL integrable vector-valued functions. The recent results in the theory of these functions obtained in [3] allow us to apply fixed point results in ordered spaces derived in [4].

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The following special types are included in the considered problems:

- differential equations and initial conditions may be implicit;
- differential equations may be singular;
- both the differential equations and the initial conditions may depend functionally on the unknown function and/or on its derivatives;
- both the differential equations and the initial conditions may contain discontinuous nonlinearities;
- problems on infinite intervals;
- problems of random type.

When E is the sequence space c_0 we obtain results for infinite systems of initial value problems, as shown in an example. Moreover, concrete finite systems are solved.

2. Preliminaries

In this section we study properties of HL integrability and differentiability of Banach space-valued functions of real variables.

A function g from a compact real interval $[a, b]$ to a Banach space E is called HL *integrable* if there is a function $f : [a, b] \to E$, called a *primitive* of g, with the following property: To each $\epsilon > 0$ there corresponds such a function $\delta : [a, b] \to (0, \infty)$ that whenever $[a, b] = \cup_i[t_{i-1}, t_i]$ and $\xi_i \in [t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1, \ldots, m$, then

$$\sum_{i=1}^{m} \|f(t_i) - f(t_{i-1}) - g(\xi_i)(t_i - t_{i-1})\| < \epsilon. \tag{2.1}$$

If g is HL integrable on $[a, b]$, it is HL integrable on every closed subinterval $[c, d]$ of $[a, b]$. The *Henstock–Kurzweil* integral of g over $[c, d]$ is defined by

$$^K\!\!\int_c^d g(s)\,\mathrm{d}s := f(d) - f(c), \quad \text{where } f \text{ is a primitive of } g.$$

The proofs for the results of the next Lemma can be found, e.g., from [5].

Lemma 2.1. (a) *The integrals of a.e. equal HL integrable functions are equal.*
(b) *Every HL integrable function is strongly measurable.*
(c) *A Bochner integrable function $u : [a, b] \to E$ is HL integrable, and*

$$\int_c^d u(s)\,\mathrm{d}s = {}^K\!\!\int_c^d u(s)\,\mathrm{d}s \quad \textit{whenever } [c, d] \textit{ is a closed subinterval of } [a, b].$$

The set $H([a, b], E)$ of all HL integrable functions $u : [a, b] \to E$ is a vector space with respect to the usual addition and scalar multiplication of functions. Identifying a.e. equal functions it follows that the space $L^1([a, b], E)$ of Bochner integrable functions from $[a, b]$ to E is a subspace of $H([a, b], E)$.

Given an open or half-open interval J of $\mathbb{R}$, denote by $H_{\mathrm{loc}}(J, E)$ the space of all strongly measurable functions $u : J \to E$ which are HL integrable on each compact subinterval of J. We assume that $H_{\mathrm{loc}}(J, E)$ is ordered a.e. pointwise, i.e.

$$u \le v \quad \text{if and only if} \quad u(s) \le v(s) \text{ for a.e. } s \in J. \tag{2.2}$$

The results of the next Lemma follow from [3], Proposition 2.1 and Lemma 2.5.

Lemma 2.2. *Let $u, v : J \to E$ be strongly measurable, $u_\pm \in H_{\mathrm{loc}}(J, E)$, and assume that $u_-(s) \le u(s) \le v(s) \le u_+(s)$ for a.e. $s \in J$. Then $u, v \in H_{\mathrm{loc}}(J, E)$, and*

$$^K\!\!\int_a^t u(s)\,\mathrm{d}s \le {}^K\!\!\int_a^t v(s)\,\mathrm{d}s \quad \textit{for all } a, t \in J,\ a \le t.$$

In our study we need the following result, which is proved in [3], Proposition 3.2.

Lemma 2.3. *Let W be a nonempty set in an order interval $[w_-, w_+]$ of $H_{\mathrm{loc}}(J, E)$.*

(a) *If W is well ordered, it contains an increasing sequence which converges a.e. pointwise to* $\sup W$.
(b) *If W is inversely well ordered, it contains a decreasing sequence which converges a.e. pointwise to* $\inf W$.

We say that a function $u : J \to E$ is *absolutely continuous in the generalized restricted sense* (ACG^*) on J if J can be expressed as such a countable union of its subsets B_n, $n \in \mathbb{N}$, that for all $\epsilon > 0$ and $n \in \mathbb{N}$ there exists such a $\delta_n > 0$ that

$$\sum_i \sup\{\|u(d) - u(c)\| : [c, d] \subseteq [c_i, d_i]\} < \epsilon$$

whenever $\{[c_i, d_i]\}$ is a finite sequence of non-overlapping intervals which have endpoints in B_n and satisfy $\sum_i (d_i - c_i) < \delta_n$.