Nonlinear Analysis 72 (2010) 4298–4303

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Local $C^1(\overline{\Omega})$-minimizers versus local $W^{1,p}(\Omega)$-minimizers of nonsmooth functionals

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ARTICLE INFO

Article history:
Received 17 December 2009
Accepted 3 February 2010

MSC:
34A60
35R70
49J52
49J53

Keywords:
Clarke's generalized gradient
Local minimizers
Nonsmooth functionals
p-Laplacian

ABSTRACT

We study not necessarily differentiable functionals of the form

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x + \frac{1}{p}\int_\Omega |u|^p \mathrm{d}x + \int_\Omega j_1(x, u)\mathrm{d}x + \int_{\partial\Omega} j_2(x, \gamma u)\mathrm{d}\sigma$$

with $1 < p < \infty$ involving locally Lipschitz functions $j_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ as well as $j_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$. We prove that local $C^1(\overline{\Omega})$-minimizers of J must be local $W^{1,p}(\Omega)$-minimizers of J.

1. Introduction

We consider the functional $J : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x + \frac{1}{p}\int_\Omega |u|^p \mathrm{d}x + \int_\Omega j_1(x, u)\mathrm{d}x + \int_{\partial\Omega} j_2(x, \gamma u)\mathrm{d}\sigma \tag{1.1}$$

with $1 < p < \infty$. The domain $\Omega \subset \mathbb{R}^N$ is supposed to be bounded with Lipschitz boundary $\partial\Omega$ and the nonlinearities $j_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ as well as $j_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are measurable in the first argument and locally Lipschitz in the second one. By $\gamma : W^{1,p}(\Omega) \to L^{q_1}(\partial\Omega)$ for $1 < q_1 < p_*$ ($p_* = (N-1)p/(N-p)$ if $p < N$ and $p_* = +\infty$ if $p \geq N$), we denote the trace operator which is known to be linear, bounded and even compact. Note that $J : W^{1,p}(\Omega) \to \mathbb{R}$ does not have to be differentiable and that it corresponds to the following elliptic inclusion

$$\begin{aligned} -\Delta_p u + |u|^{p-2}u + \partial j_1(x, u) &\ni 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \partial j_2(x, \gamma u) &\ni 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, is the negative p-Laplacian. The symbol $\frac{\partial u}{\partial \nu}$ denotes the outward pointing conormal derivative associated with $-\Delta_p$ and $\partial j_k(x, u)$, $k = 1, 2$, stands for Clarke's generalized gradient given by

$$\partial j_k(x, s) = \{\xi \in \mathbb{R} : j_k^o(x, s; r) \geq \xi r, \forall r \in \mathbb{R}\}.$$

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doi:10.1016/j.na.2010.02.006

The term $j_k^o(x, s; r)$ denotes the generalized directional derivative of the locally Lipschitz function $s \mapsto j_k(x, s)$ at s in the direction r defined by

$$j_k^o(x, s; r) = \limsup_{y \to s, t \downarrow 0} \frac{j_k(x, y + tr) - j_k(x, y)}{t},$$

(cf. [1, Chapter 2]). It is clear that $j_k^o(x, s; r) \in \mathbb{R}$ because $j_k(x, \cdot)$ is locally Lipschitz.

The main goal of this paper is the comparison of local $C^1(\overline{\Omega})$ and local $W^{1,p}(\Omega)$-minimizers. That means that if $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of J, then u_0 is also a local $W^{1,p}(\Omega)$-minimizer of J. This result is stated in our main Theorem 3.1.

Such a result was first proven for functionals corresponding to elliptic equations with Dirichlet boundary values by Brezis and Nirenberg in [2] if $p = 2$. They consider potentials of the form

$$\Phi(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \int_\Omega F(x, u),$$

where $F(x, u) = \int_0^u f(x, s)\mathrm{d}s$ with some Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$. An extension to the more general case $1 < p < \infty$ can be found in the paper of García Azorero et al. in [3]. We also refer the reader to [4] if $p > 2$. As regards nonsmooth functionals defined on $W_0^{1,p}(\Omega)$ with $2 \le p < \infty$, we point to the paper [5]. A very inspiring paper about local minimizers of potentials associated with nonlinear parametric Neumann problems was published by Motreanu et al. in [6]. Therein, the authors study the functional

$$\phi_0(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z F_0(z, x(z))\mathrm{d}z, \quad \forall x \in W_n^{1,p}(\Omega)$$

with

$$W_n^{1,p}(\Omega) = \left\{ y \in W^{1,p}(\Omega) : \frac{\partial x}{\partial n} = 0 \right\},$$

where $\frac{\partial x}{\partial n}$ is the outer normal derivative of u and $F_0(z, x) = \int_0^x f_0(z, s)\mathrm{d}s$, as well as $1 < p < \infty$. A similar result corresponding to nonsmooth functionals defined on $W_n^{1,p}(\Omega)$ for the case $2 \le p < \infty$ was proved in [7]. We also refer the reader to the paper in [8] for $1 < p < \infty$.

A recent paper about the relationship between local $C^1(\overline{\Omega})$-minimizers and local $W^{1,p}(\Omega)$-minimizers of C^1-functionals has been treated by the author in [9]. The idea of the present paper was the generalization to the more general case of nonsmooth functionals defined on $W^{1,p}(\Omega)$ with $1 < p < \infty$ involving boundary integrals which in general do not vanish.

2. Hypotheses

We suppose the following conditions on the nonsmooth potentials $j_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $j_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$.

(H1) (i) $x \mapsto j_1(x, s)$ is measurable in Ω for all $s \in \mathbb{R}$.
(ii) $s \mapsto j_1(x, s)$ is locally Lipschitz in $\mathbb{R}$ for almost all $x \in \Omega$.
(iii) There exists a constant $c_1 > 0$ such that for almost all $x \in \Omega$ and for all $\xi_1 \in \partial j_1(x, s)$ it holds that

$$|\xi_1| \le c_1(1 + |s|^{q_0 - 1}) \tag{2.1}$$

with $1 < q_0 < p^*$, where p^* is the Sobolev critical exponent

$$p^* = \begin{cases} \dfrac{Np}{N - p} & \text{if } p < N, \\ +\infty & \text{if } p \ge N. \end{cases}$$

(H2) (i) $x \mapsto j_2(x, s)$ is measurable in $\partial\Omega$ for all $s \in \mathbb{R}$.
(ii) $s \mapsto j_2(x, s)$ is locally Lipschitz in $\mathbb{R}$ for almost all $x \in \partial\Omega$.
(iii) There exists a constant $c_2 > 0$ such that for almost all $x \in \partial\Omega$ and for all $\xi_2 \in \partial j_2(x, s)$ it holds that

$$|\xi_2| \le c_2(1 + |s|^{q_1 - 1}) \tag{2.2}$$

with $1 < q_1 < p_*$, where p_* is given by

$$p_* = \begin{cases} \dfrac{(N - 1)p}{N - p} & \text{if } p < N, \\ +\infty & \text{if } p \ge N. \end{cases}$$

(iv) Let $u \in W^{1,p}(\Omega)$. Then every $\xi_3 \in \partial j_2(x, u)$ satisfies the condition

$$|\xi_3(x_1) - \xi_3(x_2)| \le L|x_1 - x_2|^\alpha,$$

for all x_1, x_2 in $\partial\Omega$ with $\alpha \in (0, 1]$.

Remark 2.1. Note that the conditions above imply that the functional $J : W^{1,p}(\Omega) \to \mathbb{R}$ is locally Lipschitz (see [10] or [11, p. 313]). That guarantees, in particular, that Clarke's generalized gradient $s \mapsto \partial J(s)$ exists.