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Generalized ε-quasi-solutions in multiobjective optimization problems: Existence results and optimality conditions

C. Gutiérrez [a], R. López [b], V. Novo [c,*]

[a] Departamento de Matemática Aplicada, E.T.S.I. Informática, Universidad de Valladolid, Edificio de Tecnologías de la Información y las Telecomunicaciones, Campus Miguel Delibes, s/n, 47011 Valladolid, Spain

[b] Departamento de Matemática y Física Aplicadas, Facultad de Ingeniería, Universidad Católica de la Ssma. Concepción, Alonso de Ribera 2850, Campus San Andrés, Casilla 297, Concepción, Chile

[c] Departamento de Matemática Aplicada, E.T.S.I. Industriales, Universidad Nacional de Educación a Distancia, c/ Juan del Rosal, 12, Ciudad Universitaria, 28040 Madrid, Spain

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ABSTRACT

This paper is concerned with approximate solutions of multiobjective optimization problems whose order cone is not necessarily the nonnegative orthant. We introduce the concept of generalized ε-quasi-solution, that extends other well-known approximate efficiency notions of the literature, and we study the limit behavior of these solutions as approximations to the efficient and weak efficient sets. Moreover, we prove several existence results and a bound for the generalized ε-quasi-solution set under convexity assumptions and by using asymptotic analysis tools. Finally, we develop optimality conditions for a particular case of these kinds of ε-quasi-solutions in nonsmooth convex and nonconvex problems.

1. Introduction

In this work we study the following multiobjective optimization problem:

$$D - \text{Min}\{f(x) : x \in C\}, \qquad (\mathcal{P})$$

where X is a Banach space, $\emptyset \neq C \subset X$ is closed, $f = (f_1, f_2, \ldots, f_m) : X \to \mathbb{R}^m$, $D \subset \mathbb{R}^m$ is a closed pointed ($D \cap (-D) = \{0\}$) convex cone and the objective space $\mathbb{R}^m$ is partially ordered via the relation $\leq_D$ defined as follows:

$$y, z \in \mathbb{R}^m, \quad y \leq_D z \iff z - y \in D.$$

In the framework of a multiobjective Pareto problem, i.e., $D = \mathbb{R}^m_+$ (the nonnegative orthant), Loridan [1, Definition 4.1] introduced the notion of ε-quasi-Pareto solution as follows.

Definition 1.1. Consider $q = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) \in \mathbb{R}^m_+ \setminus \{0\}$. $x_0 \in C$ is a q-quasi-Pareto solution if there is no $x \in C$ such that

$$f_i(x) + \varepsilon_i \|x - x_0\| \leq f_i(x_0) \quad 1 \leq i \leq m$$

with at least one strict inequality.

* Corresponding author. Tel.: +34 913986436; fax: +34 913986012.
E-mail addresses: cesargv@mat.uva.es (C. Gutiérrez), rlopez@ucsc.cl (R. López), vnovo@ind.uned.es (V. Novo).

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With this notion, Loridan extended to a multiobjective Pareto problem the ε-quasi-solution concept of a scalar optimization problem (Definition 1.1 with $m = 1$) introduced by himself in [2, Definition 3.2].

In scalar optimization, some ε-quasi-solutions are near to approximate solutions, as it is showed by Ekeland's variational principle (see [3, Theorem 1.1]).

Theorem 1.1 (*Ekeland's Variational Principle*). *In problem* $(\mathcal{P})$ *with* $m = 1$ *suppose that the objective function* f *is lower semicontinuous and consider* $\alpha > 0$, $\varepsilon > 0$ *and* $x_0 \in C$ *such that* $f(x_0) \le f(x) + \varepsilon$ *for all* $x \in C$. *Then there exists a* (ε/α)*-quasi-solution* $\hat{x} \in C$ *such that* $\|\hat{x} - x_0\| \le \alpha$.

This assertion shows that ε-quasi-solutions are very useful in practice. Indeed, when solving a scalar optimization problem via a numerical procedure, one can consider firstly ε-quasi-solutions with a small error ε, since some of them are near to feasible points whose objective value is almost optimal. Moreover, in some problems, the limit of ε-quasi-solutions when ε tends to zero is an exact solution (see, for example, [4, Remark 3.1]). From both points of view, ε-quasi-Pareto solutions are interesting too, in order to know if they are near to approximate solutions of the Pareto problem, (i.e., if they are near to some ε-efficient solution) and if they are useful to approximate the exact solution set of the problem.

Other important properties of these kinds of solutions in scalar optimization problems can be found in [5,2,4,6,7], such as existence theorems [2,5], characterizations via directional derivatives, conical approximations to the feasible set and convexity assumptions [2,4], stability results [5], multiplier rules in smooth and nonsmooth problems [2,4,6,7], saddle point conditions [2,4,6], etc. Let us observe that, in the literature, a lot of optimality conditions for approximate solutions of scalar optimization problems are essentially optimality conditions for ε-quasi-solutions (see, for example, [4, Theorems 3.1 and 3.2] and [6, Theorem 2.1]). Moreover, a lot of existence theorems and optimality conditions are established for "almost" ε-quasi-solutions, i.e., for ε-quasi-solutions of a surrogate optimization problem whose feasible set is a perturbation of the original feasible set (see, for example, [2, Theorem 5.1] and [4, Theorem 3.2]).

In multiobjective optimization problems, some results have also been obtained for ε-quasi-Pareto solutions. In [1,8,9], existence theorems were proved by combining scalarization or penalization processes with Ekeland's variational principle and in [8,9] certain multiplier rules and saddle point conditions were obtained. In [10], the notion of weak ε-quasi-Pareto solution was defined and necessary and sufficient optimality conditions for these kinds of approximate solutions were proved in nonsmooth convex Pareto problems.

Recently, in vector optimization problems, i.e., in optimization problems where the objective space is infinite dimensional, a few similar results have been proved through an ε-quasi-efficiency concept that extends Definition 1.1 in a natural way (for this notion see [11, Corollary 3.1(iii)] and [12, Definition 2.5]). To be precise, in [11, Theorems 3.2 and 3.3] and [13, Theorems 3.2 and 3.3] some multiplier rules in nonsmooth vector optimization problems with non-necessarily solid order cones have been proved by considering the Clarke and Mordukhovich subdifferentials, respectively. Moreover, in [12, Theorems 2.16 and 2.18] and [12, Section 4] necessary and sufficient conditions for the existence of these approximate solutions and saddle point results have been obtained, respectively.

However, to our knowledge, there is not any work dealing with approximate quasi-solutions in non-Pareto multiobjective optimization problems. This paper is motivated for this fact. To be precise, we focus on three questions: first, to study the behavior of ε-quasi-solutions when the precision ε tends to zero, second, to obtain existence results and a bound for the approximate quasi-solution set and third, to prove optimality conditions in multiobjective optimization problems with nonsmooth data. For these aims, a new ε-quasi-solution concept is introduced, which works with non-Pareto orders and extends various ε-efficiency notions of the literature.

The paper is structured as follows. In Section 2 some well-known definitions and results used in the sequel are recalled and the main notation is fixed. In Section 3 the concept of generalized ε-quasi-solution is defined and several properties are proved. In particular, some relations with other exact and approximate efficiency notions are showed and the Painlevé–Kuratowski limit when ε tends to zero of the ε-quasi-solution set is analyzed. In Section 4 various existence results are proved and a bound to the ε-quasi-solution set is obtained by using recession analysis tools. Finally, in Section 5 optimality conditions are proved for a particular type of ε-quasi-solutions in nonsmooth convex and nonconvex multiobjective problems by combining a scalarization process and generalized differential calculus rules.

2. Notations and preliminaries

For convenience, problem $(\mathcal{P})$ will be denoted by the pair (C, f) in some parts of this paper. As it is usual, $\mathrm{int}(H)$, $\mathrm{cl}(H)$, $\mathrm{cone}(H)$, $\mathrm{co}(H)$, $\mathrm{aff}(H)$ and H^c denote, respectively, the interior, the closure, the cone generated, the convex hull, the affine hull and the complement of a set $H \subset \mathbb{R}^m$, and we say that H is solid if $\mathrm{int}(H) \neq \emptyset$. Moreover, given a nonempty set $H \subset \mathbb{R}^m$ and $y \in \mathbb{R}^m$ we denote $d(y, H) = \inf\{\|y - z\| : z \in H\}$ and $\mathrm{B}(y, r)$ denotes the closed ball of center y and radius r in $\mathbb{R}^m$.

The following minimizer notions are considered. A point $x_0 \in C$ is said to be an efficient (resp. weak efficient) minimizer of problem $(\mathcal{P})$, denoted by $x_0 \in \mathrm{E}(\mathcal{P}, D)$ (resp. $x_0 \in \mathrm{WE}(\mathcal{P}, D)$), if

$$f(x) - f(x_0) \notin -D \setminus \{0\} \quad \forall x \in C$$

(resp. $f(x) - f(x_0) \notin -\mathrm{int}(D) \forall x \in C$). Moreover, $x_0 \in C$ is said to be a proper efficient minimizer of problem $(\mathcal{P})$, denoted by $x_0 \in \mathrm{PE}(\mathcal{P}, D)$, if there exists a solid pointed convex cone $K \subset \mathbb{R}^m$ such that $D \setminus \{0\} \subset \mathrm{int}(K)$ and $x_0 \in \mathrm{E}(\mathcal{P}, K)$.

These solution concepts are well-known and have been extensively studied in the literature (see for example [14–16]). In particular, the above proper efficiency notion is due to Henig (see [17]).