Nonlinear Analysis 72 (2010) 3247–3257

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On generalized Sundman transformation method, first integrals, symmetries and solutions of equations of Painlevé–Gambier type

Partha Guha [a,b,*], Barun Khanra [c], A. Ghose Choudhury [d]

[a] *Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, D-04103, Leipzig, Germany*
[b] *S.N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata-700098, India*
[c] *Sailendra Sircar Vidyalaya, 62A Shyampukur Street, Kolkata-700004, India*
[d] *Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Kolkata-700009, India*

ARTICLE INFO

Article history:
Received 1 October 2009
Accepted 2 December 2009

MSC:
34C14
34C20

Keywords:
Sundman transformation
Sundman symmetries
Painlevé–Gambier equations
First integrals
Jacobi equation

ABSTRACT

We employ the generalized Sundman transformation method to obtain certain new first integrals of autonomous second-order ordinary differential equations belonging to the Painlevé–Gambier classification scheme. This method not only yields systematically the known first integrals of a large number of the Painlevé–Gambier equations but also some time dependent ones, which greatly simplify the computation of their corresponding solution. In addition we also compute the Sundman symmetries of these equations.

1. Introduction

The problem of constructing solutions of a given differential equation forms the cornerstone of their analysis. Not unrelated to this problem is the issue of determining first integrals of the differential equation under consideration. This is because the existence of a sufficient number of first integrals often enables us to construct a solution by mere elimination of the derivatives of the dependent variable.

Although there are a number of well-defined methods for the solution of linear ordinary differential equations (ODEs) the same, however, cannot be said for nonlinear ODEs. It was only through the efforts of Lie towards the end of the nineteenth century that many *ad hoc* methods for the solution of nonlinear ODEs were gradually systemized. Besides it is generally acknowledged that, whenever a differential equation is amenable to a solution, it is because of some sort of underlying symmetry of the equation [1,2]. Much of Lie's work was concerned with point transformations of the form

$$(t,x) \mapsto (T,X) \quad \text{where } T = G(t,x),\ X = F(t,x)$$

with the transformation often involving one or more continuous real parameters.

Furthermore towards the very end of the nineteenth century the fact that a given differential equation could be transformed to a linear equation, that is, it could be *linearized* came to light [3]. This provided a mechanism to work out the solutions of many nonlinear differential equations by systematically transforming them to linear equations. In fact Lie

* Corresponding author at: Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, D-04103, Leipzig, Germany.
E-mail addresses: partha@bose.res.in (P. Guha), barunkhanra@rediffmail.com (B. Khanra), a_ghosechoudhury@rediffmail.com (A.G. Choudhury).

0362-546X/$ – see front matter
doi:10.1016/j.na.2009.12.004

himself [4] solved the linearization problem for second-order ordinary differential equations in the sense that he found the general form of all second-order ODEs that could be reduced to a linear equation by changing the independent and dependent variables [4].

Subsequently attempts were made to look beyond point transformations when dealing with second and higher-order ODEs, with some degree of success. For instance, one may look for nonlocal transformations under which a given ordinary differential equation is linearizable. This problem was studied by Duarté et al. [5] and considers transformations of the form

$$X(T) = F(t, x), \qquad \mathrm{d}T = G(t, x)\mathrm{d}t. \tag{1.1}$$

Here F and G are arbitrary smooth functions and it is assumed that the Jacobian $J \equiv \frac{\partial(T,X)}{\partial(t,x)} \neq 0$. If one knows the functional form of $x(t)$, then the latter transformation ceases to be nonlocal, but knowledge of $x(t)$ is what we are interested in, in the first place. Consequently (1.1) constitutes a particular type of nonlocal transformation. It must be pointed out that term nonlocal is very general in nature and it is therefore better to refer to such a transformation as a generalized Sundman transformation (GST) [6–8], in view of its similarity with the original transformation used in Sundman's analysis [9]. Other authors have also used such transformations, but have preferred to call them non point transformations [10,11].

In [5] the authors derived the most general condition under which a second-order ordinary differential equation is transformable to the linear equation

$$X''(T) = 0,$$

(here $X' = \frac{\mathrm{d}X}{\mathrm{d}T}$) under a generalized Sundman transformation. Euler and Euler, studied the case of the general anharmonic oscillator in [7], wherein they investigated the *Sundman symmetries* of second-order and third-order nonlinear ODEs. These symmetries, which are in general nonlocal transformations can be calculated systematically and can be used to find first integrals of the equations. Euler et al. used the generalized Sundman transformation to obtain a relation between a generalized Emden–Fowler equation and the first Painlevé transcendent [6].

1.1. Result and plan

In this paper we concentrate on generalized Sundman transformations and Sundman symmetries of second-order ordinary differential equations of the Painlevé–Gambier classification [12,13,18]. We compute new first integrals of some of the autonomous Painlevé–Gambier equations, which are not mentioned in the classic text by Ince [14]. The method used for this purpose is the generalized Sundman transformation. Barring the six Painlevé equations, it is known that, the remaining 44 equations of this classification scheme admit solutions in terms of known special functions. Therefore knowledge of additional (time dependent) first integrals is not essential, as far as construction of their solutions is concerned. But the deduction of time dependent first integrals is interesting from a broader perspective because of the recent interest in non autonomous ODEs. Secondly, we also compute the associated Sundman symmetries of these equations. It is true that the first integrals of the Painlevé–Gambier (PG) equations are known from other methods. In this paper we demonstrate that the first integrals of PG equations can be computed in a simple manner using Sundman's method. As a bonus we obtain the Sundman symmetries of these class of equations which are not stated elsewhere.

The *organization* of the paper is as follows. In Section 2 we introduce the notion of a generalized Sundman transformation and define the associated Sundman symmetry. Section 3 begins with a discussion of the generalized Sundman transformation for the Jacobi equation and proceeds to outline the format for its explicit evaluation. It then examines, as a special case of the Jacobi equation, particular equations of the Painlevé–Gambier classification, notably the equations numbered 11, 17, 37, 41 and 43 of Ince's book, from the viewpoints of the generalized Sundman transformation, the associated Sundman symmetry including also their solution. In Section 4, four more equations of the Painlevé–Gambier classification (namely the equations numbered 18, 19, 21 and 22) which also arise as special cases of the Jacobi equation are analyzed and their parametric solutions are constructed by exploiting the Sundman transformation.

2. Generalized Sundman transformation and symmetry

Consider an nth-order ordinary differential equation given by

$$x^{(n)} = w(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n-1)}) \tag{2.1}$$

where $x = x(t)$ and $x^{(k)} = \mathrm{d}^k x/\mathrm{d}t^k$. Such an ODE may always be written as a system of first-order ODEs

$$\dot{x^i} = w^i(\mathbf{x}) \quad i = 1, \ldots, n.$$

By a first integral we mean the following.

Definition 2.1. Let $I(\mathbf{x}, t)$ be a C^1 function on an open set $U \in \mathbb{R}^n$. Then $I(\mathbf{x}, t)$ is a first integral of the vector field $\mathbf{w} \cdot \partial_{\mathbf{x}}$ corresponding to the system of ODEs $\dot{\mathbf{x}} = \mathbf{w}(\mathbf{x})$ if and only if it is constant along any solution of the equation.

This means that given a time interval T, $I(\mathbf{x}(t), t)$ is independent of t for all $t \in T$. Formally we define a generalized Sundman transformation for (2.1) as follows.