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Existence and non-existence of global solutions of the Cauchy problem for higher order semilinear pseudo-hyperbolic equations

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ABSTRACT

We consider the Cauchy problem for a higher order pseudo-hyperbolic equation. Using the $L_p \to L_q$ type estimation for the corresponding linear problem, the existence and non-existence criteria of global solutions are found. The existence and uniqueness of smooth global solutions are also investigated. We also establish the behavior of solutions and their derivatives as $t \to +\infty$.

1. Introduction

The Cauchy problem for semilinear wave equations with a damping term

$$u_{tt} + u_t - \Delta u = f\,(u)\,, \quad (t, x) \in [0, \infty) \times R^n, \tag{1}$$

$$u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x), \quad x \in R^n \tag{2}$$

is investigated in the papers [1–21]. In [1,2], it is proved that if $f\,(u) = |u|^p$, where $1 < p \leq p_c = 1 + \frac{2}{n}$, then there exist the sufficiently small initial data $\varphi\,(x)$ and $\psi\,(x)$ for which the corresponding Cauchy problem (1), (2) has no global solutions. The existence of global solutions for problem (1), (2) is investigated in the papers [6–11,13,14].

It is well known that if $|f\,(u)| \leq c\,|u|^p$, where

$$p > p_c \quad \text{for } n = 1, 2 \tag{3}$$

and

$$2 < p \leq 3 \quad \text{for } n = 3, \tag{4}$$

then for sufficiently small φ and ψ (i.e. when $\|\varphi\|_{W_2^1(R^n)} + \|\psi\|_{L_2(R^n)}$ is sufficiently small), the corresponding Cauchy problem (1), (2) has a global solution

$$u \in C\left([0, \infty)\,; W_2^1\left(R^n\right)\right) \cap C^1\left([0, \infty)\,; L_2\left(R^n\right)\right) \quad \text{(see [6]).}$$

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For semilinear parabolic equations the question on existence and absence of global solutions are investigated in [12] (see also [4]). The number $p_c = 1 + \frac{2}{n}$ is called a Fujitas critical exponent of growth.

In [4], the different methods for investigation of non-existence of global solution for nonlinear differential equations and differential inequalities are stated in detail. The detailed references to this question are given in [4]. In this direction, we will note also the papers [7,9,13,16,17], where the global solvability of Cauchy problem of a semilinear weak damping wave equation with critical growth exponent for source term is investigated.

Only some papers studied the question on existence of global solutions for semilinear pseudo-hyperbolic and hyperbolic equations of higher order (see for example [22,23]). In the paper [23], the Cauchy problem

$$\begin{cases} u_{tt} + \Delta^2 u + u = f(u) \\ u(0,x) = \varphi(x), \qquad u_t(0,x) = \psi(x) \end{cases}$$

is considered, and under certain conditions on the growth exponent of nonlinearity $f(u)$, the existence and uniqueness of global solutions are proved.

The existence of global solution to the Cauchy problem for semilinear pseudo-hyperbolic equation

$$u_{tt} + (-1)^k \Delta^k u_{tt} + (-1)^l \Delta^l u + (-1)^k \Delta^k u_t + u_t = f(u), \quad (t,x) \in [0,\infty) \times R^n, \tag{5}$$

$$u(0,x) = \varphi(x), \qquad u_t(0,x) = \psi(x), \quad x \in R^n, \tag{6}$$

where $0 \le k \le l$

$$|f(u)| \le c\,|u|^p, \qquad \left|f'(u)\right| \le c\,|u|^{p-1}$$

and

$$p \in \left(\frac{4l}{n}, \infty\right) \quad \text{for } n < 2(l-k); \qquad p \in (2,\infty) \quad \text{for } n = 2(l-k); \tag{7}$$

$$p \in \left(\max\left\{2, \frac{4l}{n}\right\}, \frac{n}{n-2(l-k)}\right) \quad \text{for } 2(l-k) < n < 4(l-k), \tag{8}$$

is investigated in [22]. Under conditions (7), (8), existence of global solutions for "sufficiently small" initial data is proved.

In the case when $l = 1$ and $k = 0$, the conditions imposed on the nonlinearity growth in (7), (8) are more restrictive than the similar conditions (3), (4) obtained for semilinear wave equations with dissipation (see [6,22]).

In the case when $l - k = 1$, the conditions imposed on the nonlinearity growth in the paper [22] coincides with the conditions (3), (4).

In the present paper, we obtain the critical exponent question of global solvability, and also absence of global solutions for problem (5), (6). Critical exponent obtained in this paper is the number $p_c = 1 + \frac{2l}{n}$.

It is proved that if $1 < p \le p_c$, then there exist "sufficiently small" initial data for which the corresponding problem (5), (6) has no weak global solutions. If $p_c < p < \infty$ for $n < 2(l-k)$, $2 < p < \infty$ for $n = 2(l-k)$ and $2 < p \le \frac{2n}{n-2(l-k)}$ for $2(l-k) < n < 4(l-k)$, then for sufficiently small initial data there exists a global solution. If function $f(u)$ has admitted smoothness property, then for smoothness initial data existence and uniqueness of the smoothness solution are also proved. Moreover, the behavior of solutions and their derivatives as $t \to +$ are also investigated.

Note that the theory of linear pseudo-hyperbolic equations can be found in the monographs [24,25].

2. Statement of the problem and main results

We consider the Cauchy problem for semilinear pseudo-hyperbolic equations (5), (6) and assume that the following conditions are satisfied:

1_0. $0 \le k \le l,\ n < 4(l-k)$

2_0. $f(0) = 0$ and $f(u)$ satisfies the local Lipschitz condition:

$$|f(u_1) - f(u_2)| \le c\left(|u_1|^{p-1} + |u_2|^{p-1}\right)|u_1 - u_2|,$$

where $c > 0$,

$p \in \left(1 + \frac{2l}{n}, +\infty\right)$ for $n < 2(l-k)$;

$p \in (2,\infty)$ for $n = 2(l-k)$;

$p \in \left(2, \frac{n}{n-2(l-k)}\right]$ for $2(l-k) < n < 4(l-k)$.

Let us introduce the functional space

$$H_m = W_2^{l-k+m}\left(R^n\right) \times W_2^m\left(R^n\right)$$

with the inner product

$$\left\langle w^1, w^2\right\rangle_m = \int_{R^n} \nabla^{l-k+m} u^1 \nabla^{l-k+m} u^2 dx + \int_{R^n} u^1 u^2 dx + \int_{R^n} \nabla^m \upsilon^1 \nabla^m \upsilon^2 dx + \int_{R^n} \upsilon^1 \upsilon^2 dx,$$