Nonlinear Analysis 72 (2010) 3324–3336

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Generation, motion and thickness of transition layers for a nonlocal Allen–Cahn equation

Matthieu Alfaro *

Département de Mathématiques, CC051, Université Montpellier II, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France

ARTICLE INFO

Article history:
Received 20 March 2009
Accepted 8 December 2009

MSC:
35K57
45K05
35B25
35R35
53C44

Keywords:
Reaction–diffusion equation
Nonlocal PDE
Singular perturbation
Motion by mean curvature

ABSTRACT

We investigate the behavior, as $\varepsilon \to 0$, of the nonlocal Allen–Cahn equation $u_t = \Delta u + \frac{1}{\varepsilon^2} f(u, \varepsilon \int_\Omega u)$, where $f(u,0)$ is of the bistable type. Given a rather general initial datum u_0 that is independent of ε, we perform a rigorous analysis of both the generation and the motion of the interface, and obtain a new estimate for its thickness. More precisely, we show that the solution develops a steep transition layer within the time scale of order $\varepsilon^2|\ln \varepsilon|$, and that the layer obeys the law of motion that coincides with the limit problem within an error margin of order ε.

1. Introduction

This paper is concerned with the singular limit, as $\varepsilon \to 0$, of the nonlocal Allen–Cahn equation

$$(P^\varepsilon)\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2} f\left(u, \varepsilon \displaystyle\int_\Omega u\right) & \text{in } \Omega \times (0,\infty) \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$) and ν the Euclidian unit normal vector exterior to $\partial\Omega$. We assume that the nonlinearity $f(u, v)$ is smooth and that $\tilde{f}(u) := f(u, 0)$ is given by $\tilde{f}(u) := -W'(u)$, where $W(u)$ is a double-well potential with equal well depth, taking its global minimum value at $u = \pm 1$. More precisely, we assume that $\tilde{f}$ has exactly three zeros $-1 < a < 1$ such that

$$\tilde{f}'(\pm 1) < 0, \qquad \tilde{f}'(a) > 0 \quad \text{(bistable nonlinearity)}, \tag{1.1}$$

and that

$$\int_{-1}^{+1} \tilde{f}(u)\mathrm{d}u = 0. \tag{1.2}$$

* Fax: +33 0 4 67 14 93 16.
E-mail address: malfaro@math.univ-montp2.fr.

0362-546X/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2009.12.013

The condition (1.1) implies that the potential $W(u)$ attains its local minima at $u = \pm 1$, and (1.2) implies that $W(-1) = W(+1)$. In other words, the two stable zeros of $\tilde{f}$ have "balanced" stability.

Concerning the initial datum u_0, we assume its smoothness and choose $C_0 \geq 1$ such that

$$\|u_0\|_{C^0(\overline{\Omega})} + \|\nabla u_0\|_{C^0(\overline{\Omega})} + \|D^2 u_0\|_{C^0(\overline{\Omega})} \leq C_0. \tag{1.3}$$

Furthermore we define the "initial interface" Γ_0 by

$$\Gamma_0 := \{x \in \Omega | u_0(x) = a\},$$

and suppose that Γ_0 is a smooth closed hypersurface without boundary, such that, n being the Euclidian unit normal vector exterior to Γ_0,

$$\Gamma_0 \subset\subset \Omega \quad \text{and} \quad \nabla u_0(x) \neq 0 \quad \text{if } x \in \Gamma_0, \tag{1.4}$$

$$u_0 > a \quad \text{in } \Omega_0^+, \qquad u_0 < a \quad \text{in } \Omega_0^-, \tag{1.5}$$

where Ω_0^- denotes the region enclosed by Γ_0 and Ω_0^+ the region enclosed between $\partial\Omega$ and Γ_0.

Before going into more details, let us recall known facts concerning the "usual" Allen–Cahn equation, namely

$$u_t = \Delta u + \frac{1}{\varepsilon^2}\tilde{f}(u).$$

The singular limit was first studied by Allen and Cahn [1] and by Kawasaki and Ohta [2]. By using formal asymptotic arguments, they show that the limit problem, as $\varepsilon \to 0$, is a free boundary problem: the motion of the limit interface is ruled by its mean curvature. More precisely, the solution u^ε of the Allen–Cahn equation tends to a step function taking the value $+1$ on one side of a moving interface, and -1 on the other side. This sharp interface, which we will denote by Γ_t, obeys the law of motion $V_n = -\kappa$, where V_n is the normal velocity of Γ_t in the exterior direction and κ the mean curvature at each point of Γ_t.

Then, some rigorous justifications of this procedure were obtained. In the framework of classical solutions, let us mention the works of Bronsard and Kohn [3], Chen [4,5], and de Mottoni and Schatzman [6,7]. Later, in [8], the authors prove an optimal estimate for this convergence for solutions with general initial data. By performing an analysis of both the generation and the motion of interface, they show that the solution develops a steep transition layer within a very short time, and that the layer obeys the law of motion that coincides with the formal asymptotic limit $V_n = -\kappa$ within an error margin of order ε (previously, the best thickness estimate in the literature was of order $\varepsilon|\ln\varepsilon|$, [4]). For similar estimates of the thickness of the interface in related problems we refer to [9] (reaction–diffusion–convection system as a model for chemotaxis with growth), [10] (inhomogeneous Lotka–Volterra competition–diffusion system), and [11] (fully anisotropic Allen–Cahn equation).

Since the classical motion by mean curvature may develop singularities in finite time (extinction, "pinch off" phenomena, etc.), one has to define a generalized motion by mean curvature in order to study the singular limit of the Allen–Cahn equation for all time. One represents Γ_t as the level set of an auxiliary function which solves (in the viscosity sense) a nonlinear partial differential equation. This direct partial differential equation approach was developed by Evans and Spruck [12] and Chen, Giga and Goto [13]. In this framework of viscosity solutions, we refer to Evans, Soner and Souganidis [14], Barles, Soner and Souganidis [15], Barles and Souganidis [16], and Ilmanen [17] for the singular limit of reaction–diffusion equations, for all time.

We now turn back to the nonlocal Allen–Cahn equation. Problem (P^ε) was considered by Chen, Hilhorst and Logak [18]. In order to underline its relevance in population genetics and nervous transmission, they first show that (P^ε) can be seen as the limit, as $\sigma \to 0$ and $\tau \to 0$, of the FitzHugh–Nagumo system

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2} f\left(u, \varepsilon\dfrac{|\Omega|}{\gamma}v\right) \\ \tau v_t = \dfrac{1}{\sigma}\Delta v + u - \dfrac{1}{\gamma}v. \end{cases}$$

Then, they study the motion of transition layers for the solutions u^ε of (P^ε). More precisely, for "well-prepared" initial data, they prove that, as $\varepsilon \to 0$, the sharp interface limit, which we will denote by Γ_t, obeys the law of motion

$$(P^0)\begin{cases} V_n = -\kappa + c_0(|\Omega_t^+| - |\Omega_t^-|) \quad \text{on } \Gamma_t \\ \Gamma_t|_{t=0} = \Gamma_0, \end{cases}$$

where V_n is the normal velocity of Γ_t in the exterior direction, κ the mean curvature at each point of Γ_t, Ω_t^- the region enclosed by Γ_t, Ω_t^+ the region enclosed between $\partial\Omega$ and Γ_t, c_0 the constant defined by

$$c_0 = -\frac{\int_{-1}^{+1} \frac{\partial f}{\partial v}(u, 0)\mathrm{d}u}{\int_{-1}^{+1}[2(W(u) - W(-1))]^{1/2}\mathrm{d}u}, \tag{1.6}$$