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Three periodic solutions for p-Hamiltonian systems☆

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ABSTRACT

The existence of at least three periodic solutions is established for a class of p-Hamiltonian systems. Our technical approach is based on two general three critical points theorems obtained by Ricceri and Averna–Bonanno.

1. Introduction and main results

Consider the existence of periodic solutions for the p-Hamiltonian systems

$$\begin{cases} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \lambda \nabla F(t,u) + \mu \nabla G(t,u), \\ u(T) - u(0) = u'(T) - u'(0) = 0, \end{cases} \tag{1}$$

where $\lambda,\ \mu \in [0,+\infty), p > 1,\ T > 0,\ F : [0,T]\times R^N \to R$ is a function such that $F(\cdot, x)$ is continuous in $[0,T]$ for all $x \in R^N$ and $F(t,\cdot)$ is a C^1-function in R^N for almost every $t \in [0,T]$, and $G : [0,T] \times R^N \to R$ is measurable in $[0,T]$ and C^1 in R^N. $A = (a_{ij}(t))_{N\times N}$ is symmetric, $A \in C([0,T], R^{N\times N})$, and there exists a positive constant $\underline{\lambda}$ such that $(A(t)|x|^{p-2}x, x) \geq \underline{\lambda}|x|^p$ for all $x \in R^N$ and $t \in [0,T]$, that is, $A(t)$ is positive definite for all $t \in [0,T]$.

In the sequel, the Sobolev space $W_T^{1,p}$ is defined by

$$W_T^{1,p} = \left\{ u : [0,T] \to R^N \,\middle|\, \begin{array}{l} u \text{ is absolutely continuous,} \\ u(0) = u(T) \text{ and } u' \in L^p(0,T;R^N) \end{array} \right\},$$

and endowed with the norm

$$\|u\|_A = \left(\int_0^T |u'(t)|^p \mathrm{d}t + \int_0^T (A(t)|u(t)|^{p-2}u(t), u(t))\mathrm{d}t \right)^{\frac{1}{p}}.$$

Observe that

$$\begin{aligned} (A(t)|x|^{p-2}x, x) &= |x|^{p-2} \sum_{i,j=1}^N a_{ij}(t)x_i x_j \\ &\leq |x|^{p-2} \sum_{i,j=1}^N |a_{ij}(t)||x_i||x_j| \\ &\leq \left(\sum_{i,j=1}^N \|a_{ij}(t)\|_\infty \right) |x|^p, \end{aligned}$$

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then there exists a constant $\overline{\lambda} \leq \sum_{i,j=1}^{N} \|a_{ij}(t)\|_\infty$ such that $(A(t)|x|^{p-2}x, x) \leq \overline{\lambda}|x|^p$ for all $x \in R^N$. So, we have

$$\min\{1, \underline{\lambda}\}\|u\|^p \leq \|u\|_A^p \leq \max\{1, \overline{\lambda}\}\|u\|^p, \tag{2}$$

where

$$\|u\| = \left(\int_0^T |u(t)|^p \mathrm{d}t + \int_0^T |u'(t)|^p \mathrm{d}t\right)^{\frac{1}{p}}.$$

Let

$$k_0 = \sup_{u \in W_T^{1,p}\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|_A}, \quad \|u\|_\infty = \sup_{t\in[0,T]} |u(t)|, \tag{3}$$

where $|\cdot|$ is the usual norm in R^N. Since $W_T^{1,p} \hookrightarrow C^0$ is compact, one has $k_0 < +\infty$ and for each $u \in W_T^{1,p}$, there exists $\xi \in [0, T]$ such that $|u(\xi)| = \min_{t\in[0,T]} |u(t)|$. Hence, by Hölder's inequality, one has

$$\begin{aligned}
|u(t)| &= \left|\int_\xi^t u'(s)\mathrm{d}s + u(\xi)\right| \\
&\leq \int_0^T |u'(s)|\mathrm{d}s + \frac{1}{T}\int_0^T |u(\xi)|\mathrm{d}s \\
&\leq \int_0^T |u'(s)|\mathrm{d}s + \frac{1}{T}\int_0^T |u(s)|\mathrm{d}s \\
&\leq T^{\frac{1}{q}} \left(\int_0^T |u'(s)|^p \mathrm{d}s\right)^{\frac{1}{p}} + T^{-\frac{1}{p}} \left(\int_0^T |u(s)|^p \mathrm{d}s\right)^{\frac{1}{p}} \\
&\leq \max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\} \left(\left(\int_0^T |u'(s)|^p \mathrm{d}s\right)^{\frac{1}{p}} + \left(\int_0^T |u(s)|^p \mathrm{d}s\right)^{\frac{1}{p}}\right) \\
&\leq \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\} \left(\int_0^T |u'(s)|^p \mathrm{d}s + \int_0^T |u(s)|^p \mathrm{d}s\right)^{\frac{1}{p}} \\
&= \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}\|u\|
\end{aligned}$$

for each $t \in [0, T]$ and $q = \frac{p}{p-1}$. So, by (2) and the above expression, we have

$$\|u\|_\infty \leq \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}\|u\| \leq \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}(\min\{1, \underline{\lambda}\})^{\frac{-1}{p}}\|u\|_A,$$

then from this and (3) it follows that

$$k_0 \leq k := \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}(\min\{1, \underline{\lambda}\})^{\frac{-1}{p}}. \tag{4}$$

As usual, a weak solution of problem (1) is any $u \in W_T^{1,p}$ such that

$$\begin{aligned}
&\int_0^T ((|u'(t)|^{p-2}u'(t), v'(t)) + (A(t)|u(t)|^{p-2}u(t), v(t)))\mathrm{d}t \\
&\quad = \lambda\int_0^T (\nabla F(t, u(t)), v(t))\mathrm{d}t + \mu\int_0^T (\nabla G(t, u(t)), v(t))\mathrm{d}t
\end{aligned} \tag{5}$$

for all $v \in W_T^{1,p}$.

In recent years, the three critical points theorem (see [1]) of Ricceri was widely used to solve differential equations, see [2–8] and references therein. In [2–6], these authors have studied the existence of at least three weak solutions for the Dirichlet boundary value problem. In [7], Bonanno and Candito have obtained three solutions for the Neumann problem involving the p-Laplacian. In [8], the existence of at least three weak solutions has been established for a class of quasilinear elliptic systems involving the (p, q)-Laplacian with Dirichlet boundary condition.

When $p = 2$, problem (1) becomes a second order Hamiltonian system which has been extensively investigated in many papers, such as [9–11]. In [9], Bonanno and Livrea have studied the existence and multiplicity of solutions for the eigenvalue problem corresponding to the nonlinear second-order Hamiltonian systems

$$\begin{aligned}
&\ddot{u} = A(t)u - \lambda b(t)\nabla G(u), \quad t \in [0, T], \\
&u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0.
\end{aligned}$$