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Maximal monotonicity for the sum of two subdifferential operators in L^p-spaces

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This paper is dedicated to the memory of Professor Yukio Kōmura.

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ABSTRACT

This paper is devoted to providing a sufficient condition for the maximality of the sum of subdifferential operators defined on reflexive Banach spaces and proving the maximal monotonicity in $L^p(\Omega) \times L^{p'}(\Omega)$ of the nonlinear elliptic operator $u \mapsto -\Delta_m u + \beta(u(\cdot))$ with a maximal monotone graph β.

1. Introduction

Let E and E^* be a real reflexive Banach space and its dual space, respectively, and let $\phi_1, \phi_2 : E \to (-\infty, \infty]$ be proper (i.e., $\phi_1, \phi_2 \not\equiv \infty$) lower semicontinuous convex functionals with the effective domains $D(\phi_i) := \{u \in E;\ \phi_i(u) < \infty\}$ for $i = 1, 2$. Then the subdifferential operator $\partial_E \phi_i : E \to 2^{E^*}$ of ϕ_i is defined by

$$\partial_E \phi_i(u) := \left\{\xi \in E^*;\ \phi_i(v) - \phi_i(u) \ge \langle \xi, v - u\rangle_E \text{ for all } v \in D(\phi_i)\right\},$$

where $\langle \cdot, \cdot \rangle_E$ denotes the duality pairing between E and E^*, with the domain $D(\partial_E \phi_i) = \{u \in D(\phi_i);\ \partial_E \phi_i(u) \ne \emptyset\}$ for $i = 1, 2$. This paper provides a new sufficient condition for the maximal monotonicity of the sum $\partial_E \phi_1 + \partial_E \phi_2$ in $E \times E^*$ and an application to nonlinear elliptic operators in L^p-spaces.

This paper is motivated by the question of whether the following operator $\mathcal{M}$ is maximal monotone in $L^p(\Omega) \times L^{p'}(\Omega)$ with $p \in [2, \infty)$, $p' = p/(p-1)$ and a bounded domain Ω of $\mathbb{R}^N$:

$$\mathcal{M} : D(\mathcal{M}) \subset L^p(\Omega) \to L^{p'}(\Omega); \qquad u \mapsto -\Delta_m u + \beta(u(\cdot)), \tag{1}$$

where β is a maximal monotone graph in $\mathbb{R}$ such that $\beta(0) \ni 0$, and Δ_m is a modified Laplacian given by

$$\Delta_m u = \nabla \cdot \left(|\nabla u|^{m-2} \nabla u\right), \quad 1 < m < \infty$$

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equipped with the homogeneous Dirichlet boundary condition, i.e., $u|_{\partial\Omega} = 0$. The operator $\mathcal{M}$ can be divided into two parts: $u \mapsto -\Delta_m u$ and $u \mapsto \beta(u(\cdot))$, and they are maximal monotone in $L^p(\Omega) \times L^{p'}(\Omega)$. Indeed, set $E = L^p(\Omega)$ and put

$$\phi_1(u) := \begin{cases} \dfrac{1}{m}\displaystyle\int_\Omega |\nabla u(x)|^m \mathrm{d}x & \text{if } u \in W_0^{1,m}(\Omega), \\ \infty & \text{otherwise,} \end{cases} \tag{2}$$

$$\phi_2(u) := \begin{cases} \displaystyle\int_\Omega j(u(x))\mathrm{d}x & \text{if } j(u(\cdot)) \in L^1(\Omega), \\ \infty & \text{otherwise,} \end{cases} \tag{3}$$

where $j : \mathbb{R} \to (-\infty, \infty]$ is a proper lower semicontinuous convex function such that $\partial j = \beta$. Then ϕ_1 and ϕ_2 are lower semicontinuous and convex in E, and moreover, $\partial_E\phi_1(u)$ and $\partial_E\phi_2(u)$ coincide with $-\Delta_m u$ equipped with $u|_{\partial\Omega} = 0$ and $\beta(u(\cdot))$, respectively. Although every subdifferential operator is maximal monotone, the sum of two subdifferential operators might not be maximal monotone. Hence it is not obvious whether the operator $\mathcal{M} = \partial_E\phi_1 + \partial_E\phi_2$ is maximal monotone in $E \times E^*$ or not.

The maximality for the sum of two maximal monotone operators was well studied in Hilbert space settings (see [1,2]). These results were combined with nonlinear semigroup theory founded by Kōmura [3] in 1967 and developed later by Brézis and many other people for the study of nonlinear evolution equations. As for Banach space settings, a couple of sufficient conditions are proposed by Brézis et al. [4] (see also [5,6]). Let A and B be maximal monotone operators from E into E^*. Their results ensure the maximal monotonicity of $A + B$ in $E \times E^*$ if one at least of the following conditions is satisfied:

(i) $D(A) \cap (\operatorname{Int} D(B)) \neq \emptyset$,
(ii) B is dominated by A, i.e., $D(A) \subset D(B)$ and $\|B(u)\|_{E^*} \le k\|A(u)\|_{E^*} + \ell(|u|_E)$ for all $u \in D(A)$ with $k \in (0, 1)$ and a non-decreasing function ℓ in $\mathbb{R}$.

Here we write $\|C\|_{E^*} := \inf\{|c|_{E^*}; c \in C\}$ for each non-empty subset C of E^*. Furthermore, if B is a subdifferential operator, the following condition (iii) also ensures the maximal monotonicity of $A + B$, and this fact is proved in [2] for when $E = E^* = H$ is a Hilbert space; however, it can be naturally extended to a Banach space setting.

(iii) $B = \partial_E\phi$ with a proper, lower semicontinuous convex function $\phi : E \to (-\infty, +\infty]$, and

$$\phi(J_\lambda u) \le \phi(u) + C\lambda \quad \text{for } u \in D(\phi) \text{ and } \lambda > 0, \tag{4}$$

where J_λ denotes the resolvent of A in E.

Here the resolvent $J_\lambda : E \to D(A)$ is given such that $u_\lambda := J_\lambda u$ is a unique solution of $F_E(u_\lambda - u) + A(u_\lambda) \ni 0$, where F_E stands for the duality mapping between E and E^*, for each $u \in E$.

However, these results could not be applied directly to our setting for (1). As for (i), neither $D(\partial_E\phi_1)$ nor $D(\partial_E\phi_2)$ might have any interior points in E $(=L^p(\Omega))$. Condition (ii) cannot be checked unless an appropriate growth condition is imposed on β. Condition (iii) is available for the case where $p = 2$, because the duality mapping F_E of $E = L^2(\Omega)$ is the identity and the resolvent J_λ for $\partial_E\phi_2$ has a simple representation formula,

$$(J_\lambda u)(x) = (1 + \lambda\beta)^{-1}(u(x)) \quad \text{for a.e. } x \in \Omega, \tag{5}$$

which enables us to check (4). However, it is somewhat difficult to check (4) for the case where $p \neq 2$. Actually, the relation between the resolvents of $\partial_E\phi_2$ and β is unclear, since the duality mapping F_E is severely nonlinear whenever $p \neq 2$ (see (20) below).

In this paper we propose a new sufficient condition for the maximality of $\partial_E\phi_1 + \partial_E\phi_2$ in $E \times E^*$ such that the representation formula (5) in $L^2(\Omega)$ can be effectively used in applications to nonlinear elliptic operators such as (1). More precisely, we introduce a Hilbert space H as a pivot space of the triplet $E \hookrightarrow H \equiv H^* \hookrightarrow E^*$ and an extension ϕ_2^H of ϕ_2 to H, and moreover, we give a sufficient condition for the maximality in terms of the resolvent and the Yosida approximation for $\partial_H\phi_2^H$.

The treatment of the operator $\mathcal{M}$ in $L^p(\Omega)$ with $p \neq 2$ is required from recent studies on severely nonlinear problems such as generalized Allen–Cahn equations of the form

$$|u_t|^{p-2}u_t - \Delta_m u + \beta(u) + g(u) \ni f \quad \text{in } \Omega \times (0, \infty), \tag{6}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{7}$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega \tag{8}$$

with a non-monotone function $g : \mathbb{R} \to \mathbb{R}$. The main difficulty of treating (6) arises from the nonlinearity in u_t. To avoid this, one often chooses $E = L^p(\Omega)$ as a base space of analysis, since the mapping $u \mapsto |u|^{p-2}u$ from E into E^* has fine properties. Moreover, (6)–(8) can be reduced to the Cauchy problem for the following evolution equation in $E^* = L^{p'}(\Omega)$:

$$\partial_E\psi(u'(t)) + \partial_E\phi(u(t)) + g(u(\cdot, t)) \ni f(t) \quad \text{in } E^*,$$