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Linearly perturbed polyhedral normal cone mappings and applications☆

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ABSTRACT

Under a mild regularity assumption, we derive an exact formula for the Fréchet coderivative and some estimates for the Mordukhovich coderivative of the normal cone mappings of perturbed polyhedra in reflexive Banach spaces. Our focus point is a *positive linear independence condition*, which is a relaxed form of the *linear independence condition* employed recently by Henrion et al. (2010) [1], and Nam (2010) [3]. The formulae obtained allow us to get new results on solution stability of affine variational inequalities under linear perturbations. Thus, our paper develops some aspects of the work of Henrion et al. (2010) [1] Nam (2010) [3] Qui (in press) [12] and Yao and Yen (2009) [6,7].
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1. Introduction

Consider a real Banach space X with the dual denoted by X^*, a finite index set $T = \{1, 2, \ldots, m\}$, a vector system $\{a_i^* \in X^* \mid i \in T\}$, and a polyhedral convex set (a *polyhedron*, for brevity)

$$\Theta(b) = \{x \in X \mid \langle a_i^*, x\rangle \le b_i \text{ for all } i \in T\}. \tag{1.1}$$

Here $b := (b_1, \ldots, b_m) \in \mathbb{R}^m$ is a parameter. We interpret $b_1, \ldots, b_m$ as *right-hand side perturbations* of the linear inequality system

$$\langle a_i^*, x\rangle \le b_i, \quad i \in T. \tag{1.2}$$

The *active index set* corresponding to a pair (x, b), where $x \in \Theta(b)$, is defined by

$$I(x, b) = \{i \in T \mid \langle a_i^*, x\rangle = b_i\}. \tag{1.3}$$

For a subset $I \subset T$, put $\bar{I} = T \setminus I$. By b_I (resp., $b_{\bar{I}}$) we denote the vector with the components b_i where $i \in I$ (resp., $i \in \bar{I}$).

The two-variable multifunction $\mathcal{F} : X \times \mathbb{R}^m \rightrightarrows X^*$,

$$\mathcal{F}(x, b) := N(x; \Theta(b)) \quad \forall (x, b) \in X \times \mathbb{R}^m, \tag{1.4}$$

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where

$$N(x;\Theta(b)) = \begin{cases} \{x^* \in X^* \mid \langle x^*, u - x\rangle \le 0\ \forall u \in \Theta(b)\} & \text{if } x \in \Theta(b) \\ \emptyset & \text{if } x \notin \Theta(b) \end{cases}$$

represents the cone normal to $\Theta(b)$ at x in the sense of convex analysis, is said to be *the normal cone mapping* of the perturbed polyhedron $\Theta(b)$.

It is well-known [1–7] that generalized differentiability properties of $\mathcal{F}(\cdot)$ lead to useful information on solution sensitivity/stability of *variational inequalities with polyhedral convex constraint sets*. Namely, if W is Banach space and $f : X \times W \to X^*$ is a continuously Fréchet differentiable function, then the solution set of the parametric variational inequality problem

$$\text{Find } x \in \Theta(b) \quad \text{s.t. } \langle f(x, w), u - x\rangle \ge 0\ \forall u \in \Theta(b), \tag{1.5}$$

coincides with the *implicit multifunction*

$$G(w, b) := \{x \mid 0 \in f(x, w) + N(x; \Theta(b))\} \tag{1.6}$$

defined by the inclusion (called a *generalized equation*)

$$0 \in F(x, w, b) := f(x, w) + N(x; \Theta(b)). \tag{1.7}$$

As has been shown in [8–11] and the references therein, the problem of computing the Fréchet and Mordukhovich coderivatives for the implicit multifunction $(w, b) \mapsto G(w, b)$ can be reduced (in some sense) to the computation of the coderivatives of the multifunction $(x, w, b) \mapsto F(x, w, b)$. Further, since $F(x, w, b)$ is the sum of a differentiable function and the normal cone mapping $\mathcal{F}(x, b) := N(x; \Theta(b))$, it suffices to compute the coderivatives of $\mathcal{F}$ and apply the coderivative sum rules with equalities in [11, Theorem 1.62].

In this paper, under a mild regularity assumption, we derive an exact formula for the Fréchet coderivative and some estimates for the Mordukhovich coderivative of the normal cone mapping $\mathcal{F}(\cdot)$. Our focus point is a *positive linear independence condition*, which is a relaxed form of the *linear independence condition* employed recently by Henrion et al. [1], and Nam [3]. The formulae obtained allow us to get new results on solution stability of affine variational inequalities under linear perturbations. Thus, our paper develops some aspects of the work of [1,3,12,6,7].

The organization of the rest of the paper is as follows. Section 2 recalls some basic definitions of variational analysis (Fréchet and limiting normal cones for sets, Fréchet and Mordukhovich coderivatives of multifunctions), introduces the above-mentioned positive linear independence condition, and establishes two useful lemmas. The Fréchet coderivative and the Mordukhovich coderivative of $\mathcal{F}(\cdot)$ are computed, respectively, in Sections 3 and 4. The last section presents some applications of the results of Sections 3 and 4 to parametric affine variational inequalities.

2. Basic definitions and preliminaries

2.1. Basic definitions

For a multifunction $\Psi : X \rightrightarrows X^*$, the symbol $\operatorname{Lim\,sup}_{x\to\bar{x}} \Psi(x)$ denotes the *sequential Kuratowski–Painlevé upper limit* with respect to the norm topology of X and the weak* topology of X^*, i.e.,

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} \Psi(x) = \{x^* \in X^* \mid \exists \text{ sequences } x_k \to \bar{x},\ x_k^* \xrightarrow{w^*} x^*,\ \text{with } x_k^* \in \Psi(x_k) \text{ for all } k = 1, 2, \ldots\}.$$

The set $\widehat{N}_\varepsilon(\bar{x}; \Omega)$ of the Fréchet ε-normals [11] of Ω at $\bar{x} \in \Omega$ is

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) := \left\{x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \le \varepsilon\right\}, \tag{2.1}$$

where the notation $x \xrightarrow{\Omega} \bar{x}$ means $x \to \bar{x}$ and $x \in \Omega$. For $\varepsilon = 0$, the set in (2.1) is a closed convex cone which is termed the *Fréchet normal cone* of Ω at $\bar{x}$ and is denoted by $\widehat{N}(\bar{x}; \Omega)$. One puts $\widehat{N}_\varepsilon(\bar{x}; \Omega) = \emptyset$ for all $\varepsilon \ge 0$ when $\bar{x} \notin \Omega$. The cone

$$N(\bar{x}; \Omega) := \operatorname*{Lim\,sup}_{x\to\bar{x},\ \varepsilon\downarrow 0} \widehat{N}_\varepsilon(x; \Omega), \tag{2.2}$$

which is generally nonconvex and nonclosed [11, Example 1.7], is said to be the *limiting normal cone* (other names: the *basic normal cone* [11], the *Mordukhovich normal cone*) of Ω at $\bar{x}$. If $\bar{x} \notin \Omega$, then one puts $N(\bar{x}; \Omega) = \emptyset$.

If X is an Asplund space [11, Definition 2.17] then the expression on the right-hand side of (2.2) can be simplified. Namely, if X is such a space and Ω is locally closed around $\bar{x}$ then, according to [11, Theorem 2.35],

$$N(\bar{x}; \Omega) := \operatorname*{Lim\,sup}_{x\to\bar{x}} \widehat{N}(x; \Omega). \tag{2.3}$$

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