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On a numerical method for a homogeneous, nonlinear, nonlocal, elliptic boundary value problem

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ABSTRACT

In this work we develop a numerical method for the equation: $-\alpha\left(\int_0^1 u(t)\mathrm{d}t\right)u''(x)+[u(x)]^{2n+1}=0$, $x\in(0,1)$, $u(0)=a$, $u(1)=b$. We begin by establishing a priori estimates and the existence and uniqueness of the solution to the nonlinear auxiliary problem via the Schauder fixed point theorem. From this analysis, we then prove the existence and uniqueness to the problem above by defining a continuous compact mapping, utilizing the a priori estimates and the Brouwer fixed point theorem. Next, we analyze a discretization of the above problem and show that a solution to the nonlinear difference problem exists and is unique and that the numerical procedure converges with error $\mathcal{O}(h)$. We conclude with some examples of the numerical process.

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1. Introduction

In [1] the authors analyzed the analytic and numerical solution to the nonlocal elliptic B.V.P.:

$$-\alpha\left(\int_0^1 u(t)\mathrm{d}t\right)u''(x)=f(x),\quad x\in(0,1),\qquad u(0)=a,\qquad u(1)=b,$$

which was a one-dimensional problem similar to those discussed in [2–5]. The crux of the research in [1] essentially amounted to analyzing conditions on the coefficient and data, which lead to the existence and uniqueness of the solution, the existence and uniqueness of a numerical approximation and the convergence of the numerical approximation to the analytic solution.

This analysis also motivated the consideration of the following equation:

$$-\alpha\left(\int_0^1 u(t)\mathrm{d}t\right)u''(x)+[u(x)]^{2n+1}=0,\quad x\in(0,1),\qquad u(0)=a,\qquad u(1)=b,\tag{1.1}$$

which is a nonlinear, nonlocal boundary value problem whose u'' coefficient is dependent upon the integral of the solution over the domain of the solution, analogous to the nonlocal problem in [1]. We ultimately show that the solution of the analytic problem (1.1) exists and is unique, and also demonstrate that a suitable numerical approximation yields similar existence and uniqueness results as well. In addition, we establish that an approximation to (1.1) converges to the analytic

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solution of (1.1) with a satisfactory degree of accuracy, as was discovered for the problem defined in [1], and also supply examples of the numerical process.

It is also important to mention that the numerical methods developed in this paper function as a paradigm for further numerical methods to be developed for various other analytic problems with physical applications. In fact, much research has been done on nonlocal modeling problems with structures very similar to (1.1). Below we discuss some of these problems and their applications. In each example, we also display the modeling equation to emphasize the similarity with (1.1).

A nonlocal problem modeling Ohmic heating with variable thermal conductivity was studied in [6], including an analysis of the asymptotic behavior and the blow-up of solutions. This model has the following form:

$$\begin{aligned}
&u_t = (u^3 u_x)_x + \frac{\lambda f(u)}{\left(\int_{-1}^{1} f(u)\mathrm{d}x\right)^2}, \quad -1 < x < 1,\ t > 0,\\
&u(-1,t) = u(1,t) = 0, \quad t > 0\\
&u(x,0) = u_0(x), \quad -1 < x < 1,
\end{aligned}$$

where λ is a positive parameter. This work was motivated by the problem studied in [7,8] that models Ohmic heating, which contains a standard linear diffusion term of the form u_{xx} and is also a nonlocal problem.

In addition, Stańczy [9] studied nonlocal elliptic equations that arise in physical models including systems of particles in thermodynamical equilibrium interacting via gravitational (Coulomb) potential and a similar problem was studied in [10]. The equations that Stańczy studied also arose in fully turbulent behavior of a real flow, thermal runway in Ohmic heating, shear bands in metals deformed under high strain rates and one dimensional fluid flows with rate of strain proportional to a power of stress multiplied by a function of temperature. The model in [9] is defined as follows:

$$-\Delta\varphi = M\frac{f(\varphi)^\alpha}{\left(\int_\Omega f(\varphi)\right)^\beta} \quad \text{in } \Omega \in \mathbb{R}^n,$$

where $\varphi = 0$ on $\partial\Omega$.

Lastly, a nonlocal problem that arises as a local model for the temperature in a thin region, which occurs during linear friction welding, was studied in [11]. A similar model was addressed in [12], which models thermo–viscoelastic flows. The model in [11] has the structure seen below:

$$u_t = u_{xx} + f(u)\left(\int_0^\infty f(u)\mathrm{d}y\right)^{-(1+a)} \quad \text{for } 0 < x < \infty,$$

with $u_x = 0$ on $x = 0$ and $u_x \to -1$ as $x \to \infty$.

2. A priori estimates for the solution of the nonlinear auxiliary equation

We first assume $\alpha := \alpha(q)$ is a continuous positive function defined over $-\infty < q < \infty$ and bounded below by a positive real constant α_0. We define the nonlinear auxiliary problem as

$$-\alpha(q)u''(x) + [u(x)]^{2n+1} = 0, \quad x \in (0,1), \qquad u(0) = a, \qquad u(1) = b, \tag{2.1}$$

where a and b are positive real constants. We then have the following theorem.

Theorem 2.1. *For $u := u(x)$ a solution of (2.1) we have $0 < u \le \max(a, b)$.*

Proof. First assume $u > \max(a, b)$. Then, there must exist a number, say $x_0 \in (0, 1)$, so that $u(x_0)$ is a positive maximum. Therefore,

$$-\alpha(q)u''(x_0) + [u(x_0)]^{2n+1} > 0,$$

which contradicts (2.1). Hence, $u \le \max(a, b)$.

To show $u > 0$ we consider the operator

$$\mathcal{L}(\phi) := -\alpha(q)\phi''(x) + [u(x)]^{2n}\phi(x)$$

and the function

$$\xi(x) := u(x) - \min(a, b)\mathrm{e}^{-\delta x}, \tag{2.2}$$

where δ is a real constant. Now assume $\xi < 0$. Then there must exist a negative minimum of ξ, at say $\check{x}_0$, where $\check{x}_0 \in (0, 1)$. Consequently, at $\check{x}_0$

$$\mathcal{L}(\xi)(\check{x}_0) = -\alpha(q)\xi''(\check{x}_0) + [u(\check{x}_0)]^{2n}\xi(\check{x}_0) \le 0. \tag{2.3}$$