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Uniqueness and nondegeneracy of the ground state for a quasilinear Schrödinger equation with a small parameter

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ABSTRACT

We study least energy solutions of a quasilinear Schrödinger equation with a small parameter. We prove that the ground state is nondegenerate and unique up to translations and phase shifts using bifurcation theory.

1. Introduction

In this paper we consider the following quasilinear Schrödinger equation:

$$\begin{aligned} &\mathrm{i}\partial_t\phi(t,x)+\Delta\phi(t,x)+\lambda\phi(t,x)\Delta|\phi(t,x)|^2+|\phi(t,x)|^{p-1}\phi(t,x)=0 \quad (t,x)\in(0,\infty)\times\mathbf{R}^N,\\ &\phi(0,x)=\phi_0(x)\quad x\in\mathbf{R}^N, \end{aligned} \tag{1}$$

where i is the imaginary unit, $N \geq 1$, $p > 1$ and $\phi : \mathbf{R}^N \to \mathbf{C}$. This equation appears in different physical models, such as the superfluid film equation in plasma physics. We refer the reader to [1] for a more detailed bibliography on the physical background. The mathematical theory for this equation is still not well established, even as regards the short time dynamics. The main difficulty which arises in the study of the Cauchy problem for this equation is the presence of the quasilinear term, which causes the phenomenon called *loss of derivatives*. To overcome this problem, one asks for high regularity for the initial datum and then the local well-posedness can be proved just in Sobolev spaces of high order. In particular there are no local well-posedness results for the energy space (see (4)) and so a Gagliardo–Nirenberg type inequality (which is available, as proved in [1]) cannot guarantee global well-posedness. For the main results in this direction we refer the reader to Colin [2], Colin et al. [1], Kenig et al. [3], Lange [4], Poppenberg [5], and Poppenberg et al. [6]. Despite still not having a satisfactory theory of local well-posedness, mathematicians have been able to prove the existence of a special class of time periodic solutions in the form $\phi(t,x) = u(x)\mathrm{e}^{\mathrm{i}\omega t}$ called *standing waves*, representing particle at rest. Here $u : \mathbf{R}^N \to \mathbf{C}$ solves the quasilinear elliptic equation

$$-\Delta u-\lambda u\Delta|u|^2+\omega u-|u|^{p-1}u=0, \tag{2}$$

while $\omega > 0$ is the time–frequency region. Eq. (2) is variational and is the Euler–Lagrange equation of the associated energy functional $\mathcal{E}^{\lambda}_{\omega}$ which is

$$\mathcal{E}^{\lambda}_{\omega}(u)=\frac{1}{2}\int_{\mathbf{R}^N}|\nabla u|^2\mathrm{d}x+\frac{\lambda}{4}\int_{\mathbf{R}^N}|\nabla|u|^2|^2\mathrm{d}x+\frac{\omega}{2}\int_{\mathbf{R}^N}|u|^2\mathrm{d}x-\frac{1}{p+1}\int_{\mathbf{R}^N}|u|^{p+1}\mathrm{d}x. \tag{3}$$

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The natural space in which this functional is well-defined (see [1,7]) is the following:

$$X_{\mathbf{C}} = \left\{u \in H^1(\mathbf{R}^N, \mathbf{C}) : \int_{\mathbf{R}^N} |u|^2 |\nabla |u||^2 \mathrm{d}x < \infty\right\}. \tag{4}$$

Thanks to the variational structure, solutions of Eq. (2) can be found by means of critical point theory. An important role is played by least energy solutions which are commonly called *ground states*. Their main feature is that they minimize the energy, namely they satisfy

$$\mathcal{E}_\omega(u) = m_\omega^\lambda, \tag{5}$$

where

$$m_\omega^\lambda = \inf\{\mathcal{E}_\omega(u) : u \text{ is a nontrivial weak solution of (2)}\}.$$

We denote by $\mathcal{G}_\omega^\lambda$ the set of weak solutions to (2) satisfying (5). Some authors have studied solutions belonging to the set $\mathcal{G}_\omega^\lambda$; see for example [8]. More results exist for not necessarily least energy solutions: we mention the works by Colin and Jeanjean [9], Liu et al. [10] and Liu and Wang [11]. The most complete result (to our knowledge) concerning ground states of (2) is due to Colin et al. [1] (in the case $\lambda = 1$) who proved that ground states exist, are real and positive (up to phase shifts) classical solutions, decay exponentially at infinity with their first and second derivatives and moreover are radial with respect to some point. For the precise statement of their result we refer the reader to Section 2, Theorem 2.2.

The first result of the present paper answers the problem of uniqueness for λ small enough, which was left open in [1].

Theorem 1.1. *Let $1 < p < 2^*$. There exists $\bar{\lambda}$ such that for $0 < \lambda < \bar{\lambda}$ there exists only one real, positive, radial and exponentially decaying ground state $u_\lambda(x)$ of Eq. (2). Moreover suppose $v \in \mathcal{G}_\omega^\lambda$; then there exists $\xi \in \mathbf{R}^N$ and $\theta \in [0, 2\pi]$ such that $v_\lambda(x) = u_\lambda(x - \xi)e^{i\theta}$, where $u_\lambda(x)$ is the only real, positive, radial and exponentially decaying ground state of (2).*

A natural question which arises when one deals with ground states is that of their orbital stability or instability. In [1] the authors proved that for $p > 3 + \frac{4}{N}$ ground states are unstable to blow-up, while they give only partial results for the complementary case. A first step towards the complete answer is understanding whether the ground state u_λ is nondegenerate, which means that the kernel of the linearized operator around u_λ associated with Eq. (2) is entirely spanned by the infinitesimal generators of the symmetries of the equation. As regards this issue we have the following result.

Theorem 1.2. *Let $1 < p < 2^*$. There exists $\bar{\lambda}$ (the same as in Theorem 1.1) such that for $0 < \lambda < \bar{\lambda}$ the ground state u_λ of Eq. (2) is nondegenerate in the sense of Definition 1.3.*

Here is our definition of nondegeneracy for u_λ.

Definition 1.3. The ground state u_λ of (2) is nondegenerate if and only if the following properties hold:

- (ND) $\ker[I''(u_\lambda)] = \left\{iu_\lambda(x), \frac{\partial u_\lambda(x)}{\partial x_j}, j = 1, \ldots, N\right\}$;
- (Fr) $I''(u_\lambda)$ is an index 0 Fredholm map.

The proving of these facts is carried out through bifurcation theory. It is not the first time that bifurcation theory has been used to study Eq. (2): we refer the reader to Liu et al. [12] and to Liu and Wang [13] for related results. However here we do not need their results thanks to the change of variable presented in Section 2.1. The range of p is not the optimal one in either of the theorems; for more details we refer the reader to Remark 2.8 at the end of the paper.

2. Proofs of the main theorems

2.1. Existence of the ground state for fixed λ

The existence result is mainly a consequence of the works of Colin and Jeanjean [9] and Colin et al. [1]. Looking for a ground state of (2) and following the strategy of [9,1] we use a change of unknown, $u = f_\lambda(v)$, where f_λ is a solution of the following Cauchy problem:

$$f_\lambda'(t) = \frac{1}{\sqrt{1 + 2\lambda f_\lambda^2(t)}}, \quad f_\lambda(0) = 0. \tag{6}$$

We have the following lemma:

Lemma 2.1. 1. *f_λ is uniquely defined, smooth and invertible;*
2. *$|f_\lambda'| \leq 1$ for $t \in \mathbf{R}$;*
3. *$\frac{f_\lambda(t)}{t} \to 0$ as $t \to 0$;*
4. *$\frac{f_\lambda(t)}{\sqrt{t}} \to 2^{\frac{1}{4}}\lambda^{\frac{1}{2}}$.*